OPTIMAL ACCELERATED LIFE TEST DESIGNS

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Dedicated to the Loving

Memory of my Father

CERTIFICATE

This is to certify that the contents of this thesis entitled **"OPTIMAL ACCELERATED LIFE TEST DESIGNS"** is the original research work of **Mr. Nesar Ahmad** carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinance and regulations of Aligarh Muslim University, Aligarh.

I further certify that the work of this thesis, either partially or fully, has not been submitted to any other University or Institution for the award of any other degree or diploma.

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PREFACE

The reliability theory has been formulated as the science of prediction, estimating or optimizing the probability of survival, the mean life or, more generally, the life distribution of components or system. Product reliability contribute much to quality and competitiveness. Much management and engineering effort goes into evaluating reliability, assessing new desings and design and manufacturing changes, identifying causes of failure, and comparing designs, vendors, materials, manufacturing methods, and the like. Major decisions are based on life test data, often from a few units. Moreover, many products last so long that life testing at design cnditions is impractical. Many products can be life tested at high stress conditions to yield failures quickly. Analyses of data from such an accelerated test yield needed information on product life at design conditions. Accelerated life tests are being increasingly employed to obtain information about the life distribution of materials or products in shorter span of time.

The test units are induced to early failures by applying higher than usual levels of stress. The test data obtained at accelerated conditions are then extrapolated to the design stress level to estimate the life distribution.

In this thesis I have developed an optimal accelerated life test designs for some life time distributions such as Rayleign, Weibull, and Burr Type XII distributions. The thesis comprises of five chapters. A brief summary of the problem is presented at the beginning of each chapters, and corresponding computational results in the form of tables are provided at the end of the chapter 3, chapter 4, and chapter 5.

Chapter 1 is an introductory part of this thesis, which describes some fundamental aspects of an accelerated life test including important parametric models and test plans.

Chapter 2 presents a brief survey of the development of research work in Accelerated life Test (ALT) designs in chronological order. Sixteen landmark research contribution have been selected for given a thorough insight into the ALT designs problems and solutions, which covered almost entire spectrum of the optimum test plans.

Chapter 3 introduces statistically optimal accelerated life test plans under the assumptions of periodic inspection and Type-I censoring, for the case of Rayleigh failure distribution. For optimal plans, the high stress level is standardized, the proportion of test units allocated, low stress level and the inspection times for equally spaced (ES) are determined such that the asymptotic variance of the maximum likelihood estimator of p^{th} quantile at the use condition is minimized. Sensitivity analyses and procedures for selecting a sample size are also discussed.

Chapter 4 deals with the case where the lifetime at a stress level follow Weibull distribution. Optimal accelerated life test plans are developed under the assumptions of periodic inspection and type I censoring. The inspection times and low stress level are determined such that the asymptotic variance of the maximum likelihood estimator of the estimated mean or p^{th} quantile at the use condition is minimized. Sensitivity analysis is also carried out to see how sensitive the asymptotic variance of the estimated mean is with respect to errors involved in the guessed failure probabilities at the normal and high stress levels.

Chapter 5 presents designs of Optimal Accelerated Life Tests for the Burr Type-XII distribution under Periodic Inspection and Type I censoring. The life distribution has the distinction that its particular case contain log-logistic distribution for $m = 1$. It is assumed that a log-linear relationship exists between the scale parameter of the lifetime distribution and the stress. For optimal plans, the proportion of test units are allocated, the high stress level is standardized, low test stress level and inspection times for equally spaced (ES) are determined, which minimize the asymptotic variance (AsVar) of the maximum likelihood estimator (MLE) of the log mean life or q^{th} quantile of the lifetime distribution at design (use) stress.

Computational results for different values of the shape parameters m and δ, indicate that the asymptotic variance of the MLE of the log mean life or q^{th} quantile of the lifetime distribution at the design stress is not sensitive to the number of inspections at overstress levels. Sensitivity analysis is also conducted for different values of m and δ to assess the uncertainties involved in the guessed failure probabilities for the unknown parameters at the design and high stress levels. Procedures for selecting a sample size and guidelines for planning an accelerated life test are also discussed.

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CHAPTER 1

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CHAPTER 1

FUNDAMENTALS OF ACCELERATED TESTING, MODELS AND TEST PLANS

1.1 INTRODUCTION

This chapter presents an introduction t0 accelerated testing methods, models and Test plans. Section 1.2 describes various important aspects of planning an accelerated life test. Section 1.3 describes mathematical models for accelerated life tests with constant stress, consisting of lifetime distributions and life-stress relationships. These models are essential background for subsequent chapters. All planning and data analyses for accelerated tests are based on such models. The model depends on the product, the test method, the accelerating stress, the form of the specimen, and other factors. Section 1.4 gives maximum likelihood (ML) methods, which are basic for analyzing censored (and complete) data. This section pertains only to constant-stress tests and data with a single failure mode. Section 1.5 describes various accelerated life test plans. It also gives the accuracy (standard errors) of estimates for such plans and guidance on how many specimens to test. Presented here are optimal, traditional, and good compromise plans. Optimal plans yield the most accurate estimates of life at the design stress. Traditional plans consist of equally spaced test stress levels each with the same number of specimens. Good compromise plans run more specimens at low stress than at high stress.

1.2 ELEMENTS OF ACCELERATED LIFE TEST PLANS

This section presents background on accelerated test data and common types of acceleration of tests (i.e., high usage rate, overstress, censoring and degradation) and stress loading.

1.2.1 TYPES OF DATA

Accelerated test data can be divided into two types. Namely, the product characteristic of interest is (1) life or is (2) some other measure of performance, such as tensile strength or ductility.

The following paragraphs describe the common types of life data from a single test or design condition.

Complete. Complete data consist of the exact life (failure age) of each sample unit.

Censored. Often when life data are analyzed, some units are unfailed, and their failure times are known only to be beyond their present running times. Such data are said to be censored on the right. In older literature, such data or tests are called truncated. Unfailed units are called run-outs, survivors, removals, and suspensions. Such censored data arise when some units are (1) removed from test or service before they fail, (2) still running at the time of the data analysis, or (3) removed from test or service because they failed from an extraneous cause such as test equipment failure. Similarly, a failure time known only to be before a certain time is said to be censored on the left. If all unfailed units have a common running time and all failure times are earlier, the data are said to be singly censored on the right. Singly censored data arise when units are started together at a test condition and the data are analyzed before all units fail. Such data are singly time censored if the censoring time is fixed; then the number of failures in that fixed time is random. Time censored data are also called Type I censored. Data are singly failure censored if the test is stopped when a specified number of failures occurs. The time to that fixed number of failures is random. Time censoring is more common in practice. Failure censoring is more common in the theoretical literature, as it is mathematically more tractable.

Multiply censored. Much data censored on the right have differing running times intermixed with the failure times. Such data are called multiply censored (also progressively, hyper-, and arbitrarily censored).

Competing modes. A mix of competing failure modes occurs when sample units fail from different causes.

1.2.2 TYPES OF ACCELERATION

The following paragraphs describe the common types of acceleration. They include high usage rate, overstress and degradation.

High Usage Rate

A simple way to accelerate the life of many products is to run the product at a higher usage rate. The following are two common ways of doing such compressed time testing.

i) **Faster.** One way to accelerate is to run the product faster. For example, in many life tests, rolling bearings run at about three times their normal speed. High usage rate may also be used in combination with overstress testing. For example, such bearings are also tested under higher than normal mechanical load. Another example of high usage rate involves a voltage endurance test of an electrical insulation by Johnston and others (1979).

ii) **Reduced off time.** Many products are off much of the time in actual use. Such products can be accelerated by running them a greater fraction of the time.

Overstress Testing

Overstress testing consists of running a product at higher than normal levels of some accelerating stress(es) to shorten product life or to degrade product performance faster. Typical accelerating stresses are temperature, voltage, mechanical load, thermal cycling, humidity, and vibration.

Degradation

Accelerated degradation testing involves overstress testing. Instead of life, product performance is observed as it degrades over time.

1.2.3 TYPES OF STRESS LOADING and ACCELERATING STRESSES

The stress loading in an accelerated test can be applied various ways. Descriptions of common stress loadings follow. They include constant, cyclic, step and progressive stress loading.

Constant stress. The most common stress loading is constant stress. Each specimen is run at a constant stress level.

Step stress. In step-stress loading, a specimen is subjected to successively higher levels of stress. A specimen is first subjected to a specified constant stress for a specified length of time. If it does not fail, it is subjected to a higher stress level for a specified time. The stress on a specimen is thus increased step by step until it fails.

Progressive stress. In progressive stress loading, a specimen undergoes a continuously increasing level of stress. Different groups of specimens may undergo different progressive stress patterns.

Cyclic stress. In use, some products repeatedly undergo a cyclic stress loading. For example, many metal components repeatedly undergo a mechanical stress cycle. A cyclic stress test for such a product repeatedly loads a specimen with the same stress pattern at high stress levels.

Accelerating Stresses

In practice, one must decide how to accelerate a test. Should one use high temperature, mechanical load, voltage, current, vibration, humidity, or whatever? Should one use a combnination of stresses?

Standard stresses. For many products there are standard test methods and accelerating stresses. For example, high temperature and voltage are usually used to accelerate life tests of electrical insulation and electronics. Such standard methods and stresses are usually based on much engineering experience.

No standard stresses. For other products there may be no standard stresses. Then the responsible engineers need to determine suitable accelerating stresses. Experimental work to determine appropriate stresses may be required.

Multiple stresses. In some cases more than one accelerating stress may be used, for various reasons.

1.3 MODELS FOR ACCELERATED LIFE TESTS

A statistical model for an accelerated life test consists of (1) a life distribution that represents the scatter in product life and (2) a relationship between "life" and stress. Usually the mean (and sometimes the standard deviation) of the life distribution is expressed as a function of the accelerating stress.

This section presents the commonly used life distributions such as the exponential, Weibull, Rayleigh, log-logistic, Burr Type XII, Extreme value, normal and lognormal distributions, and also presents basic concepts for life distributions, including the reliability function and hazard function. This section also present life-stress relationships. The most widely used basic relationships are (1) the Arrhenius relationship for temperature-accelerated tests and (2) the inverse power relationship. Singpurwalla (1975) surveys a number of models.

1.3.1 EXPONENTIAL DISTRIBUTION

Exponential cumulative distribution function. The population fraction failing by age t is

$$F(t) = 1 - e^{-t/\theta}, \quad t \geq 0 . \qquad (1.3.1.1)$$

$\theta > 0$ is the mean time to failure (MTTF). θ is in the same measurement units as t, for example, hours, months, cycles, etc. Its failure rate is defined as

$$\lambda \equiv 1/\theta \qquad (1.3.1.2)$$

and is a constant.

Exponential reliability. The population fraction surviving age t is

$$R(t) = e^{-t/\theta}, \quad t \geq 0 . \qquad (1.3.1.3)$$

Exponential percentile. The $100P^{th}$ percentile is

$$\tau_P = -\theta \ln(1-P) . \qquad (1.3.1.4)$$

Exponential probability density. Differentiation of (1.3.1.1) yields

$$f(t) = (1/\theta)\, e^{-t/\theta} , \quad t \geq 0 . \qquad (1.3.1.5)$$

Also,

$$f(t) = \lambda\, e^{-\lambda t} , \quad t \geq 0 . \qquad (1.3.1.6)$$

Exponential mean. The mean is

$$E(T) = \int_0^\infty t(1/\theta)\, e^{-t/\theta}\, dt = \theta . \qquad (1.3.1.7)$$

This shows why θ is called the mean time to failure (MTTF).

Exponential variance. For an exponential distribution,

$$\mathrm{Var}(T) = \int_0^\infty t^2(1/\theta)\exp(-t/\theta)\,dt - \theta^2 = \theta^2 \qquad (1.3.1.8)$$

This is the square of the mean.

Exponential standard deviation. For an exponential distribution,

$$\sigma(T) = (\theta^2)^{1/2} = \theta \ . \qquad (1.3.1.9)$$

This equals the mean.

Exponential Hazard Function. The exponential hazard function is

$$h(t) = 1/\theta = \lambda \ , \quad t \geq 0 \ . \qquad (1.3.1.10)$$

Only the exponential distribution has a constant failure rate, a key characteristic.

1.3.2 WEIBULL DISTRIBUTION

The Weibull distribution is often used for product life, because it models either increasing or decreasing failure rates simply. It is used to describe the life of roller bearings, electronic components, ceramics, capacitors, and dielectrics in accelerated tests.

Weibull cumulative distribution. The population fraction failing by age t is

$$F(t) = 1 - \exp[-(t/\alpha)^{\beta}] \ , \quad t > 0 \ . \qquad (1.3.2.1)$$

The shape parameter β and the scale parameter α are positive. α is also called the characteristic life. α has the same units as t, for example, hours, months, cycles, etc. β is a unitless pure number.

Weibull probability density. For a Weibull distribution,

$$f(t) = (\beta/\alpha^{\beta})\ t^{\beta-1} \exp[-(t/\alpha)^{\beta}] \ , \quad t > 0 \ . \qquad (1.3.2.2)$$

β determines the spread in log life; high (low) β corresponds to small (great) spread. For $\beta = 1$, the Weibull distribution is the exponential distribution.

Weibull reliability function. The population fraction surviving age t is

$$R(t) = \exp[-(t/\alpha)^{\beta}] \ , \quad t > 0 \ . \qquad (1.3.2.3)$$

Weibull percentile. The $100P^{th}$ percentile of a Weibull distribution is

$$\tau_P = \alpha[-\ln(1 - P)]^{1/\beta} \ . \qquad (1.3.2.4)$$

Weibull hazard function. For a Weibull distribution,

$$h(t) = (\beta/\alpha)\ (t/\alpha)^{\beta-1} \ , \quad t > 0 \ . \qquad (1.3.2.5)$$

A power function of time, h(t) increases for $\beta > 1$ and decreases for $\beta < 1$. For $\beta = 1$, the failure rate is constant.

1.3.3 RAYLEIGH DISTRIBUTION

Rayleigh cumulative distribution. The population fraction failing by age t is

$$F(t) = 1 - \exp[-(t^2/2\theta^2)] \ , \quad t, \theta > 0 \ . \qquad (1.3.3.1)$$

θ is also called the characteristic life. θ has the same units as t, for example, hours, months, cycles, etc.

Rayleigh probability function. The probability density function is

$$f(t) = (t/\theta^2) \exp[-(t^2/2\theta^2)] , \quad t, \theta > 0 . \qquad (1.3.3.2)$$

Rayleigh reliability function. The population fraction surviving age t is

$$R(t) = \exp[-(t^2/2\theta^2)] , \quad t, \theta > 0 . \qquad (1.3.3.3)$$

Rayleigh percentile. The $100P^{th}$ percentile of a Rayleigh distribution is

$$\tau_P = \theta[-\ln(1 - P)]^{1/2} . \qquad (1.3.3.4)$$

Rayleigh hazard function. The hazard function is

$$h(t) = t/\theta^2 , \quad t > 0 . \qquad (1.3.3.5)$$

It is a linear function of time. Thus the Rayleigh distribution would be especially suitable for life testing experiments of components which age with time.

1.3.4 LOG-LOGISTIC DISTRIBUTION

Log-logistic cumulative distribution. The population fraction failing by age t is

$$F(t) = [1 + (t/\theta)^{\delta}]^{-1} , \quad t, \delta, \theta > 0 . \qquad (1.3.4.1)$$

θ is in the same measurement units as t, for example, hours, months, cycles, etc. In terms of $\lambda = 1/\theta$,

$$F(t) = 1 - [1 + (\lambda t)^{\delta}]^{-1} , \quad t > 0 . \qquad (1.3.4.2)$$

Log-logistic probability density. It is given by

$$f(t) = \frac{\delta}{\theta} (t/\theta)^{\delta-1} [1 + (t/\theta)^{\delta}]^{-2} , \quad t > 0 . \qquad (1.3.4.3)$$

Although this model has been used occasionally in life testing applications, it has the advantage (like the Weibull and exponential models) of having simple algebraic expressions for the survivor and hazard function. It is therefore more convenient in handling censored data than the lognormal distribution while providing a good approximation to it except in the extreme tails.

Also,

$$f(t) = \lambda\delta(\lambda t)^{\delta-1}[1 + (\lambda t)^{\delta}]^{-2} , \quad t > 0 . \qquad (1.3.4.4)$$

Log-logistic reliability function. The population fraction surviving age t is

$$R(t) = [1 + (t/\theta)^{\delta}]^{-1} , \quad t > 0 . \qquad (1.3.4.5)$$

Log-logistic percentile. The $100P^{th}$ percentile is

$$\tau_P = \theta\left(P/(1-P)\right)^{1/\delta} . \qquad (1.3.4.6)$$

Log-logistic hazard function. For a log-logistic distribution,

$$h(t) = \frac{\delta/\theta . (t/\theta)^{\delta-1}}{1 + (t/\theta)^{\delta}} , \quad t > 0 . \qquad (1.3.4.7)$$

1.3.5 BURR TYPE XII DISTRIBUTION

Burr Type XII cumulative distribution. The population fraction failing by age t is

$$F(t) = 1 - \frac{1}{[1 + (t/\theta)^{\delta}]^{m}} , \quad t > 0 . \qquad (1.3.5.1)$$

The shape parameters δ and m and the scale parameter θ are positive. θ is also called the characteristic life and it has the same units as t. Its failure rate is defined as $\lambda \equiv 1/\theta$. In terms of λ,

$$F(t) = 1 - \frac{1}{[1 + (\lambda t)^{\delta}]^{m}}, \quad t > 0 . \qquad (1.3.5.2)$$

Burr Type XII probability density. The probability function

$$f(t) = \frac{m(\delta/\theta).(t/\theta)^{\delta-1}}{[1 + (t/\theta)^{\delta}]^{m+1}}, \quad t > 0 , \qquad (1.3.5.3)$$

is unimodal if $\delta > 1$, and L-shaped if $\delta = 1$. For $m = 1$, the Burr Type XII distribution is the log-logistic distribution. Also,

$$f(t) = \frac{m(\lambda\delta)(\lambda t)^{\delta-1}}{[1 + (\lambda t)^{\delta}]^{m+1}}, \quad t > 0 . \qquad (1.3.5.4)$$

Burr Type XII reliability function. The population fraction surviving age t is

$$R(t) = \frac{1}{[1 + (t/\theta)^{\delta}]^{m}}, \quad t > 0 . \qquad (1.3.5.5)$$

Burr Type XII percentile. The $100P^{th}$ percentile is

$$\tau_P = \theta \left[\frac{1 - (1 - P)^{1/m}}{(1 - P)^{1/m}} \right]^{1/\delta} . \qquad (1.3.5.6)$$

Burr Type XII hazard function. For a Burr Type XII distribution, the hazard function is

$$h(t) = \frac{m(\delta/\theta)(t/\theta)^{\delta-1}}{1 + (t/\theta)^{\delta}}, \quad t > 0 . \qquad (1.3.5.7)$$

This hazard function is identical to the Weibull hazard function aside from the denominator factor $1 + (t/\theta)^{\delta}$; it is monotone decreasing from ∞ if $\delta < 1$ and is monotone decreasing from $\lambda = 1/\theta$ if $\delta = 1$. If $\delta > 1$, the hazard function resembles the lognormal hazard function in that it increases from zero to a maximum at $t = \theta(\delta - 1)^{1/\delta}$ and decreases towards zero thereafter.

1.3.6 EXTREME VALUE DISTRIBUTION

The (smallest) extreme value distribution is needed background for analytic methods for Weibull data. Indeed the (base e) log of time to failure for a Weibull distribution has an extreme value distribution. Like the Weibull distribution, the smallest extreme value distribution may be suitable for a "weakest link" product.

Extreme value cumulative distribution. The population fraction below y is

$$F(y) = 1 - \exp\{-\exp[(y-\xi)/\delta]\}, \; -\infty < y < \infty . \qquad (1.3.6.1)$$

The location parameter ξ may have any value from $-\infty$ to ∞. The scale parameter δ is positive, and it determines the spread of the distribution. ξ and δ are in the same units as y.

Extreme value density. For an extreme value distribution,

$$f(y) = (1/\delta)\exp[(y-\xi)/\delta] \; \exp\{-\exp[(y-\xi)/\delta]\}, \qquad (1.3.6.2)$$
$$-\infty < y < \infty .$$

Extreme value reliability function. For an extreme value distribution,

$$R(y) = \exp\{-\exp[(y-\xi)/\delta]\}, \; -\infty < y < \infty . \qquad (1.3.6.3)$$

Extreme value percentile. The $100P^{th}$ percentile is

$$\eta_P = \xi + u_P\delta \; ; \qquad (1.3.6.4)$$

here

$$u_P \equiv \ln[-\ln(1 - P)] \qquad (1.3.6.5)$$

is the $100P^{th}$ standard extreme value percentile ($\xi = 0$ and $\delta = 1$).

Extreme value hazard function. For an extreme value distribution,

$$h(y) = (1/\delta)\exp[(y - \xi)/\delta] \; , \; -\infty < y < \infty \; . \qquad (1.3.6.6)$$

1.3.7 NORMAL DISTRIBUTION

Normal cumulative distribution function. The population fraction failing by age y is

$$F(y) = \int_{-\infty}^{y} (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right] dx \; , \qquad (1.3.7.1)$$

$$-\infty < y < \infty \; .$$

μ is the population mean and may have any value. σ is the population standard deviation and must be positive. μ and σ are in the same measurement units as y. (1.3.7.1) can be expressed in terms of the standard normal cumulative distribution function $\Phi()$ as

$$F(y) = \Phi[(y - \mu)/\sigma] \; , \quad -\infty < y < \infty \; . \qquad (1.3.7.2)$$

$\Phi()$ is (1.3.7.1) evaluated at $\mu = 0$ and $\sigma = 1$.

Normal probability density The probability density is

$$f(y) = (2\pi\sigma^2)^{-1/2} \exp[-(y - \mu)^2/(2\sigma^2)] \; , \qquad (1.3.7.3)$$

$$-\infty < y < \infty \; .$$

Normal percentile. The $100P^{th}$ percentile is

$$\eta_P = \mu + z_P\sigma \ ; \qquad (1.3.7.4)$$

z_P is the $100P^{th}$ standard normal percentile.

Normal mean and standard deviation. For the normal distribution,

$$E(Y) = \mu \quad \text{and} \quad \sigma(Y) = \sigma \qquad (1.3.7.5)$$

are the distribution parameters.

1.3.8 LOGNORMAL DISTRIBUTION

The lognormal distribution is widely used for life data, including metal fatigue, solid state components (semiconductors, diodes, GaAs FETs, etc.), and electrical insulation. The lognormal and normal distributions are related; this fact is used to analyze lognormal data with methods for normal data.

Lognormal cumulative distribution. The population fraction failing by age t is

$$F(t) = \Phi\{[\log(t) - \mu]/\sigma \ , \quad t > 0 \ . \qquad (1.3.8.1)$$

μ is called the log mean and may have any value from $-\infty$ to ∞. σ is called the log standard deviation and must be positive. μ and σ are not "times" like t. The cumulative distribution can also be written as

$$F(t) = \Phi\{[\log(t/\tau_{.50})]/\sigma\} = \Phi\{\log[t/\tau_{.50})^{1/\sigma}]\} \ ; \qquad (1.3.8.2)$$

here $\tau_{.50} = \log(\mu)$ is the median. (1.3.8.2) is similar to the Weibull cumulative distribution.

Lognormal probability density. For a lognormal distribution,

$$f(t) = \{0.4343/[(2\pi)^{1/2}t\sigma]\} \cdot \exp\{-[\log(t)-\mu]^2/(2\sigma^2)\}\ , \quad t > 0\ . \qquad (1.3.8.3)$$

Percentile. The $100P^{th}$ lognormal percentile is

$$\tau_P = \text{antilog}[\mu + z_P\sigma] = 10^{\mu+z_P\sigma}\ ; \qquad (1.3.8.4)$$

here z_P is the $100P^{th}$ standard normal percentile.

Lognormal reliability function. The population fraction surviving age t is

$$R(t) = 1 - \Phi\{[\log(t)-\mu]/\sigma\} = \Phi\{-[\log(t)-\mu]/\sigma\}\ . \qquad (1.3.8.5)$$

1.3.9 ARRHENIUS LIFE-TEMPERATURE RELATIONSHIP

This section presents the Arrhenius law for reaction rates and motivates the Arrhenius life relationship.

The Arrhenius life relationship below describes the life of products and test specimens that run under constant temperature.

The Relationship

This section motivates the Arrhenius relationship.

Arrhenius law. According to the Arrhenius rate law, the rate of a simple (first-order) chemical reaction depends on temperature as follows

$$\text{rate} = A'\exp[-E/(kT)]\ ; \qquad (1.3.9.1)$$

E is the activation energy of the reaction, usually in electron-volts.
k is Boltzmann's constant.
T is the absolute Kelvin temperature.
A' is a constant that is characteristic of the product failure mechanism and test conditions.

The rate of metal diffusion is described by the same equation. thus the following Arrhenius life relationship based on (1.3.9.1) may describe failures due to diffusion in solid state devices and certain other products made of metal.

Motivation. The product is assumed to fail when some critical amount of the chemical has reacted (or diffused); a simple view of this is

(time to failure) = (critical amount)/rate.

This suggests that nominal time τ to failure ("life") is inversely proportional to the rate (1.3.9.1). This yields the Arrhenius life relationship

$$\tau = A \exp[E/(kT)] \quad . \qquad (1.3.9.2)$$

Linearized relationship. The (base 10) logarithm of (1.3.9.2) is

$$\log(\tau) = \gamma_0 + (\gamma_1/T) \qquad (1.3.9.3)$$

where

$$\gamma_1 = \log(e)(E/k) \cong 0.4343E/k \quad . \qquad (1.3.9.4)$$

Thus the log of "nominal life", $\log(\tau)$, is a linear function of inverse absolute temperature $x = 1/T$. Common choices are the 50^{th}, 63.2^{th}, and 10^{th} percentiles. (1.3.9.4) can be expressed as

$$E = 2.303\ k\ \gamma_1 \qquad (1.3.9.5)$$

Arrhenius acceleration factor. By (1.3.9.2) the Arrhenius acceleration factor between life τ at temperature T and life τ' at reference temperature T' is

$$K = \tau/\tau' = \exp\{(E/k)[(1/T)-(1/T')]\} \quad . \qquad (1.3.9.6)$$

1.3.10 INVERSE POWER RELATIONSHIP

This relationship is sometimes called the inverse power law or simply the power law.

The power relationship below describes the life of products and test specimens that run under constant stress.

The Relationship

Suppose that the accelerating stress variable V is positive. The inverse power relationship (or law) between "nominal" life τ of a product and V is

$$\tau(V) = A/V^{\gamma_1} \; ; \qquad (1.3.10.1)$$

here A and γ_1 are parameters characteristic of the product and the test method. Equivalent forms are

$$\tau(V) = (A'/V)^{\gamma_1} \text{ and } \tau(V) = A''(V_0/V)^{\gamma_1} \; ;$$

here V_0 is a specified (standard) level of stress. The parameter γ_1 is called the power or exponent.

Linearized relationship. The natural logarithm of (1.3.10.1) is

$$\ln(\tau) = \gamma_0 + \gamma_1[-\ln(V)] \; . \qquad (1.3.10.2)$$

Thus the log of "typical life", $\ln(\tau)$, is a linear function of the transformed stress $x = -\ln(V)$. "Life" τ is usually taken to be a specified percentile of the life distribution.

Power acceleration factor. By (1.3.10.1), the power acceleration factor between life τ at stress V and life τ' at reference stress V' is

$$K = \tau/\tau' = (V'/V)^{\gamma_1} \; . \qquad (1.3.10.3)$$

1.3.11 MULTIVARIABLE RELATIONSHIPS

This section presents relationship between life and two or more variables, which may be stress or other predictor variables.

This section first presents a general multivariable relationship - the log-linear relationship; it includes the generalized Eyring relationship.

Log-linear Relationship

A general, simple relationship for "nominal" life τ (say, a percentile) is the log-linear relationship

$$\ln(\tau) = \gamma_0 + \gamma_1 x_1 + \ldots + \gamma_j x_j \ . \qquad (1.3.11.1)$$

Here γ_0, γ_1, ... , γ_j are coefficients characteristic of the product and test method; they are usually estimated from data. x_1, x_2, ... , x_j are (possibly transformed) variables. Any x_j may be a function (transformation) of one or any number of basic engineering (predictor or independent) variables. (1.3.11.1) is used in parametric analyses with an assumed form of the life distribution. It is also used in nonparametric analyses without an assumed form of the life distribution.

Generalized Eyring Relationship

The generalized Eyring relationship has been used to describe accelerated life tests with temperature and one other variable. Relationship to express "nominal" product life τ as a function of absolute temperature T and a (possibly transformed) variable V, it is

$$\tau = (A/T)\exp[B/(kT)] \times \exp\{V[C + (D/kT)]\} \ . \qquad (1.3.11.2)$$

Here A, B, C and D are coefficients to be estimated from data, and k is Boltzmann's constant. (1.3.11.2) is equivalent to (1.3.11.1) where

$$\tau' = \tau.T \text{ is "life"}, \quad \gamma_0 = \ln(A), \ \gamma_1 = B/k, \ x_1 = 1/T \ ,$$
$$\gamma_2 = C, \ x_2 = V, \quad \gamma_3 = D/k, \ x_3 = V/T.$$

$x_3 = x_1 x_2 = V(1/T)$ is an "interaction term" for $x_1 = (1/T)$ and $x_2 = V$.

Logistic Regression Relationship

The logistic regression relationship is widely used in biomedical applications where the dependent variable is binary; that is, it is in one of two mutually exclusive categories, for example, dead or alive. The logistic relationship for the proportion p in a particular category (say, "failed") as a function of J independent variables $x_1, \ldots, x_J$ is

$$\ln[(1-p)/p] = \gamma_0 + \gamma_1 x_1 + \ldots + \gamma_J x_J \; ; \qquad (1.3.11.3)$$

here $\gamma_0, \gamma_1, \ldots, \gamma_J$ are unknown coefficients to be estimated from data.

In accelerated testing, (1.3.11.3) might be used when the life data are quantal-response data; that is, each specimen is inspected once to determine whether it has failed by its inspection age. Then p is the fraction failed, and one of the independent variables is (log) age at inspection.

1.4 MAXIMUM LIKELIHOOD METHODS

This section presents maximum likelihood (ML) theory and calculations for fitting a model to observed, censored, and interval data. These calculations yield ML estimates.

1.4.1 THE MODEL

The statistical distribution. The dependent variable y is assumed to have a specified continuous cumulative distribution function

$$F(y;\theta_1, \ldots, \theta_Q) \; ; \qquad (1.4.1.1)$$

here $\theta_1, \ldots, \theta_Q$ are the Q distribution parameters. The probability density is

$$f(y;\theta_1, \ \ldots \ ,\theta_Q) \equiv dF(y;\theta_1, \ \ldots \ ,\theta_Q)/dy \ . \qquad (1.4.1.2)$$

Relationships for the distribution parameters. Each distribution parameter is expressed as an assumed function of the J independent variables $x_1, \ \ldots \ ,x_j$ and the P distinct model coefficients $\gamma_1, \ \ldots \ ,\gamma_P$; that is,

$$\begin{aligned} \theta_1 &= \theta_1(x_1, \ \ldots \ ,x_j; \ \gamma_1, \ \ldots \ ,\gamma_P) \ , \\ &\vdots \\ \theta_Q &= \theta_1(x_1, \ \ldots \ ,x_j; \ \gamma_1, \ \ldots \ ,\gamma_P) \ . \end{aligned} \qquad (1.4.1.3)$$

The functional form of each relationship is specified (assumed), but the values of the model coefficients are unknown. They are estimated from the test data.

1.4.2 LIKELIHOOD FUNCTION

Suppose specimen i has an observed value y_i of the dependent variable. Then its likelihood is

$$L_i = f(y_i; \ \theta_{1i}, \ \ldots \ ,\theta_{Qi}) \ ; \qquad (1.4.2.1)$$

here $\theta_{1i}, \ \ldots \ ,\theta_{Qi}$ are the specimen's parameter values. this L_i is the "probability" of an observed failure at y_i.

Right censored. Suppose specimen i has the dependent variable censored on the right at y_i; that is, its value is above y_i. Its likelihood is

$$L_i = 1 - F(y_i; \ \theta_{1i}, \ \ldots \ ,\theta_{Qi}) \ ; \qquad (1.4.2.2)$$

This L_i is the probability that the specimen's life is above y_i.

Left censored. Suppose specimen i has the dependent variable censored on the left at y_i; that is, its value is below y_i. Its likelihood is

$$L_i = F(y_i; \ \theta_{1i}, \ \ldots \ ,\theta_{Qi}) \ . \qquad (1.4.2.3)$$

This L_i is the probability that the specimen's life is below y_i.

Interval. Suppose specimen i has a dependent variable value that is known only to be in an interval with end points $y_i < y_i'$. Then its likelihood is

$$L_i = F(y_i'; \theta_{1i}, \ldots, \theta_{Qi}) - F(y_i; \theta_{1i}, \ldots, \theta_{Qi}). \quad (1.4.2.4)$$

This L_i is the probability that the specimen's life is in the interval (y_i, y_i').

Sample likelihood. It is assumed that the n sample specimens have statistically independent random variations in their values of the dependent variable. Then,

$$L \equiv L_1 \times L_2 \times \ldots \times L_n . \quad (1.4.2.4)$$

This sample likelihood is the joint probability of the n dependent variable outcomes.

Log likelihood. The log likelihood of specimen i is

$$£_i \equiv \ln(L_i) . \quad (1.4.2.5)$$

The sample log likelihood is

$$£ \equiv \ln(L) = £_1 + £_2 + \ldots + £_n . \quad (1.4.2.6)$$

where £ and $£_i$ are the functions of $\gamma_1, \ldots, \gamma_p$.

1.4.3 ML ESTIMATES OF MODEL COEFFICIENTS

Estimates. The maximum likelihood estimates $\hat{\gamma}_1, \ldots, \hat{\gamma}_p$ of $\gamma_1, \ldots, \gamma_p$ are the coefficient values that maximize the sample log likelihood over the allowed ranges of $\gamma_1, \ldots, \gamma_p$.

Likelihood equations. The values $\hat{\gamma}_1, \ldots, \hat{\gamma}_p$ that maximize $£(\gamma_1, \ldots, \gamma_p)$ can be found by the usual calculus method.

Namely, set equal to zero the P derivatives of $\pounds(\gamma_1, \ldots, \gamma_P)$ with respect to $\gamma_1, \ldots, \gamma_P$, and solve the following likelihood equations for $\hat{\gamma}_1, \ldots, \hat{\gamma}_P$:

$$\begin{aligned} \partial\pounds(\gamma_1, \ldots, \gamma_P)/\partial\gamma_1 &= 0 , \\ &\vdots \\ \partial\pounds(\gamma_1, \ldots, \gamma_P)/\partial\gamma_P &= 0 . \end{aligned} \tag{1.4.3.1}$$

Usually these nonlinear equations in $\gamma_1, \ldots, \gamma_P$ cannot be solved algebraically. Then they must be solved with numerical methods.

1.4.4 FISHER AND COVARIANCE MATRICES

After obtaining $\hat{\gamma}_1, \ldots, \hat{\gamma}_P$, one estimates their covariance matrix as described next. This matrix is calculated from the Fisher matrix.

Fisher matrix. The local Fisher information matrix is the P×P symmetric matrix of negative second partial derivatives

$$\mathbf{F} = \begin{bmatrix} -\partial^2\hat{\pounds}/\partial\gamma_1^2 & -\partial^2\hat{\pounds}/\partial\gamma_1\partial\gamma_2 & \cdots & -\partial^2\hat{\pounds}/\partial\gamma_1\partial\gamma_P \\ -\partial^2\hat{\pounds}/\partial\gamma_2\gamma_1 & -\partial^2\hat{\pounds}/\partial\gamma_2^2 & \cdots & -\partial^2\hat{\pounds}/\partial\gamma_2\partial\gamma_P \\ \vdots & \vdots & \vdots & \vdots \\ -\partial^2\hat{\pounds}/\partial\gamma_P\gamma_1 & -\partial^2\hat{\pounds}/\partial\gamma_P\gamma_2 & \cdots & -\partial^2\hat{\pounds}/\partial\gamma_P^2 \end{bmatrix} . \tag{1.4.4.1}$$

The caret ^ indicates that the derivative is evaluated at $\gamma_1 = \hat{\gamma}_1, \ldots, \gamma_P = \hat{\gamma}_P$.

Covariance matrix. The inverse of **F** is the local estimate **V** of the (asymptotic) covariance matrix for $\hat{\gamma}_1, \ldots, \hat{\gamma}_P$. That is,

$$
\mathbf{V} = \begin{bmatrix} \mathrm{var}(\hat{\gamma}_1) & \mathrm{cov}(\hat{\gamma}_1, \hat{\gamma}_2) & \cdots & \mathrm{cov}(\hat{\gamma}_1, \hat{\gamma}_p) \\ \mathrm{cov}(\hat{\gamma}_2, \hat{\gamma}_1) & \mathrm{var}(\hat{\gamma}_2) & \cdots & \mathrm{cov}(\hat{\gamma}_2, \hat{\gamma}_p) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{cov}(\hat{\gamma}_p, \hat{\gamma}_1) & \mathrm{cov}(\hat{\gamma}_p, \hat{\gamma}_2) & \cdots & \mathrm{var}(\hat{\gamma}_p) \end{bmatrix} . \qquad (1.4.4.2)
$$

Standard error. The standard error $\sigma(\hat{\gamma}_p)$ of $\hat{\gamma}_p$ is the standard deviation of asymptotic normal distribution. The estimate of $\sigma(\hat{\gamma}_p)$ is

$$
s(\hat{\gamma}_p) = [\mathrm{var}(\hat{\gamma}_p)]^{1/2} . \qquad (1.4.4.3)
$$

1.4.5 ESTIMATE OF A FUNCTION AND ITS VARIANCE

Estimate. Often one wants to estimate the value of a given function $h = h(\gamma_1, \ldots, \gamma_p)$ of the model coefficients. The ML estimate $\hat{h}$ of h is

$$
\hat{h} = h(\hat{\gamma}_1, \ldots, \hat{\gamma}_p) . \qquad (1.4.5.1)
$$

Variance. The estimate of variance of $\hat{h}$ is calculated as follows. Calculate the column vector of partial derivatives $\partial h/\partial \gamma_p$:

$$
\hat{\mathbf{H}} = \begin{bmatrix} \partial\hat{h}/\partial\gamma_1 \\ \vdots \\ \partial\hat{h}/\partial\gamma_p \end{bmatrix} . \qquad (1.4.5.2)
$$

The local estimate of the (asymptotic) variance of $\hat{h}$ is

$$
\begin{aligned}
\mathrm{var}(\hat{h}) &= \hat{\mathbf{H}}'\mathbf{V}\hat{\mathbf{H}} \qquad (1.4.5.3) \\
&= \sum_{p=1}^{p} (\partial\hat{h}/\partial\gamma_p)^2 \, \mathrm{var}(\hat{\gamma}_p) + 2\sum_{p<p'}\sum (\partial\hat{h}/\partial\gamma_p)(\partial\hat{h}/\partial\gamma_{p'})\mathrm{cov}(\gamma_p, \gamma_{p'}) ;
\end{aligned}
$$

here $\hat{H}'$ is the transpose of $\hat{H}$ and is a row vector. the estimate $s(\hat{h})$ of the standard error of $\hat{h}$ is

$$s(\hat{h}) = [var(\hat{h})]^{1/2} \qquad (1.4.5.4)$$

1.5 TEST PLANS

1.5.1 PLANS FOR THE SIMPLE MODEL AND COMPLETE DATA

This section shows how to evaluate the accuracy of a plan and how to determine sample size.

Model. The assumptions of the simple linear-lognormal model are:

1. Specimen life t at any stress level has a lognormal distribution.
2. The standard deviation σ of log life is constant.
3. The mean log life at a (positively transformed stress x is

$$\mu(x) = \gamma_0 + \gamma_1 x \ . \qquad (1.5.1.1)$$

Here γ_0, γ_1, and σ are parameters to be estimated from data.

Estimates. Suppose n specimens are tested at stress levels $x_1, x_2, \ldots, x_n$ (some may be equal), and the observed log lives are $y_1, y_2, \ldots, y_n$. Then,

$$c_1 = \left[\sum y_i(x_i-\bar{x})\right] / \left[\sum (x_i-\bar{x})^2\right] \ , \ c_0 = \bar{y} - c_1\bar{x} \ ; \qquad (1.5.1.2)$$

The least squares estimate of the mean log life $\mu(x_0)$ at a specified x_0 (say, design stress) is

$$m(x_0) = c_0 + c_1 x_0 \ . \qquad (1.5.1.3)$$

Accuracy. The variance of $m(x_0)$ is

$$\mathrm{Var}[m(x_0)] = \{1 + (x_0 - \bar{x})^2[n/\sum(x_i - \bar{x})^2]\}\sigma^2/n \,. \qquad (1.5.1.4)$$

Test constraints. If the test stresses $x_1, \ldots, x_n$ are unconstrained (1.5.4.1) is minimized by running all specimens at x_0; that is $x_1 = \ldots = x_n = x_0$. Then the test is not accelerated and is usually much too long. Thus it is necessary to choose a lowest allowed test stress level x_L, which produces long life. This is chosen as low as possible to minimize. The highest allowed test stress x_H, which produces short life, is chosen as high as possible to minimize (1.5.1.4).

The Optimum Plan

Optimum stress levels. The optimum stress levels are the minimum and maximum in thy allowed test range. Intermediate levels are not used.

Extrapolation factor. For a stress level x,

$$\xi \equiv (x_H - x)/(x_H - x_L) \;; \qquad (1.5.1.5)$$

here x_L is the lowest allowed stress level and x_H is the highest one.

Optimum allocation. A derivation of the optimum allocation of n test specimens follows. Let p denote the proportion tested at $x_L(\xi = 1)$; $1 - p$ are tested at $x_H(\xi = 0)$. Thus the $\xi_i \equiv (x_H - x_i)/(x_H - x_L)$ values of the specimens are $\xi_1 = \ldots = \xi_{pn} = 1$ and $\xi_{pn+1} = \ldots = \xi_n = 0$, and

$$\mathrm{Var}[m(x_0)] = \left[1 + \frac{(\xi_0-p)^2}{p(1-p)}\right]\sigma^2/n \,, \qquad (1.5.1.6)$$

since $\sum\xi_i = np$, $\bar{\xi} = p$, and $\sum\xi_i^2 = np$. The fraction p^* that minimizes (1.5.1.6) is

$$p^* = \xi_0/(2\xi_0 - 1) \ . \qquad (1.5.1.7)$$

The minimum variance is

$$\mathrm{Var}_2^*[m(x_0)] = [1 + 4\xi_0(\xi_0 - 1)]\sigma^2/n \ . \qquad (1.5.1.8)$$

The number of specimens allocated to the low test stress is the integer nearer to np^*. Such rounding results in a variance slightly larger than (1.5.1.8). This allocation also minimizes the variance of the least squares estimate of the $100p^{th}$ percentile of log life $y_p(x_0) = m(x_0) + z_p s$.

Extreme allocations. Two extreme cases of (1.5.1.8) are informative. First, if the lowest test stress equals the design stress, then $\xi_0 = 1$ and $p^* = 1$. Second, if x_0 is much below x_L, then $p^* = 1/2$. That is, the specimens are allocated equally to x_L and x_H.

Traditional Plans

A commonly used test plan has equally spaced test stress levels and equal numbers of specimens at each. Such traditional plans with equal allocation of specimens to two, three, and four stress levels are described here for comparison with better plans.

Two stress levels. For two levels with equal allocation (n/2 specimens at each level), $\xi_1 = \ldots = \xi_{(n+2)+1} = \ldots = \xi_n = 0$. Then

$$\mathrm{Var}_2[m(x_0)] = [1 + 4(\xi_0 - 0.5)^2]\sigma^2/n \ , \qquad (1.5.1.9)$$

Three stress levels. For three equally spaced levels with equal allocation (n/3 specimens at each level), $\xi_1 = \ldots = \xi_{n/3} = 1$, $\xi_{(n/3)+1} = \ldots = \xi_{2n/3} \cong 1/2$, $\xi_{(2n/3)+1} = \ldots = \xi_n = 0$. Then

$$\mathrm{Var}_3[m(x_0)] = [1 + 6(\xi_0 - 0.5)^2]\sigma^2/n \, , \qquad (1.5.1.10)$$

Four stress levels. For four equally spaced levels with equal allocation (n/4 specimens at each level), $\xi_1 = \ldots = \xi_{n/4} = 1$, $\xi_{(n/4)+1} = \ldots = \xi_{n/2} \cong 2/3$, $\xi_{(n/2)+1} = \ldots = \xi_{3n/4} \cong 1/3$, and $\xi_{(3n/4)+1} = \ldots = \xi_n = 0$. Then

$$\mathrm{Var}_4[m(x_0)] = [1 + (36/5)(\xi_0 - 0.5)^2]\sigma^2/n \, , \qquad (1.5.1.11)$$

Good Test Plans

A good test plan should be multi-purpose and robust and provide accurate estimates. Such a plan consists of three or four equally spaced test levels with unequal allocation. such unequal allocation puts more specimens at the extremes of the test range and fewer in the middle. Of course, more specimens at the lowest test level results in longer test time until all specimens fail. The time to complete the test can be controlled in part by the choice of the lowest test stress level.

Sample Size

One can determine the number n that achieves a desired accuracy of the estimate of the mean log life or median life at design stress.

To determine n, one can specify that the estimate $m(x_0)$ is to be within $\pm w$ of the true $\mu(x_0)$ with high probability γ. For any test plan, the n that achieves this is (1.5.1.4) rewritten as

$$n = \{1 + (x_0 - x)^2[n/\sum(x - \bar{x})^2]\}(K_\gamma\sigma/w)^2 \, . \qquad (1.5.1.12)$$

1.5.2 PLANS FOR THE SIMPLE MODEL AND SINGLY CENSORED DATA

Test Method. It is assumed that

1. Each test unit runs a specified test time τ (the censoring time) if it does not fail sooner.
2. The highest test stress x_H is specified.

3. The specified design stress x_0 is below the test stresses.

The test time τ should be to minimize the variance of estimates from the test. The highest test stress x_H should be as high as possible. This minimizes the standard error of the estimate of any percentile at the design stress.

Model. The assumptions are:

1. At any stress, life has a lognormal distribution.
2. The standard deviation σ of log life is a constant.
3. The mean log life is a linear function of a stress x:

$$\mu(x) = \gamma_0 + \gamma_1 x \ . \tag{1.5.2.1}$$

The model parameters γ_0, γ_1, and σ are to be estimated from test data.

Then the 100Pth percentile of life $\tau_P(x)$ or log life $\eta_P(x)$ at a stress x is

$$\eta_P(x) = \log[\tau_P(x)] = \mu(x) + z_P\sigma = \gamma_0 + \gamma_1 x + z_P\sigma \ ; \tag{1.5.2.2}$$

Optimization criterion. Here an optimum test plan minimizes the variance (or standard error) of the ML estimate of the median life at a specified (design) stress x_0. The estimate of another percentile (1.5.2.2) could be optimized; this would require a different plan.

Best Traditional Plans

Traditional plans have K equally spaced test stresses, each with the same number of test units. The highest test stress x_H must be specified. The "best" plan uses a lowest test stress x_L that minimizes the standard error of the ML estimate of the log mean at a specified design stress x_0.

Lowest test stress. The best lowest test stress x_L is given by Nelson and Kielpinski (1976) as

$$x_L = x_H + \xi_K(x_0 - x_H) \quad . \tag{1.5.2.3}$$

Standard error. For such a best plan with n specimens, the large-sample standard error of teh ML estimate $\hat{\mu}_0 = \hat{\gamma}_0 + \hat{\gamma}_1 x_0$ of $\mu_0 = \gamma_0 + \gamma_1 x_0$ is

$$\sigma(\hat{\mu}_0) = \sigma(V_K/n)^{1/2} \quad ; \tag{1.5.2.3}$$

here V_K depends on K, a, and b.

Sample size. One requires that, with a desired high probability $\hat{\gamma}$, $\hat{\mu}_0$ fall within ±w of the true μ_0. The sample size n_K that achieves this is approximately

$$n_K \cong V_K(K_\gamma \sigma/w)^2 \tag{1.5.2.4}$$

Optimum Test Plans

It is assumed that the high test stress x_H is specified. The low test stress x_L and its proportion of the specimens are chosen to minimize the standard error of the ML estimate $\hat{\mu}_0$ of the median at a specified stress x_0.

Optimum stress. The optimum low test stress is given by Nelson and Kielpinski (1976) as

$$x_L^* = x_H + \xi^*(x_0 - x_H) \quad ; \tag{1.5.2.5}$$

here ξ^* is a function of a and b.

Standard error. For an optimum plan with n specimens, the large-sample standard error of $\hat{\mu}_0$ is

$$\sigma(\hat{\mu}_0) = \sigma(V^*/n)^{1/2} \quad . \tag{1.5.2.5}$$

CHAPTER 2

ACCELERATED LIFE TEST PLANS

2.1 SURVEY OF THE RESEARCH WORK

This chapter presents a brief survey of the development of research work in Accelerated Life Test (ALT) Designs in chronological order. Sixteen landmark research contributions have been selected for giving a thorough insight into the ALT designs problems and solutions, which covered almost entire spectrum of the optimum test plans. They are Chernoff (1962); Kielpinski and Nelson (1975); Meeker and Nelson (1975); Nelson and Kielpinski (1976);Nelson and Meeker (1978); DeGroot and Goel (1979); Miller and Nelson (1983); Meeker (1984); Meeker (1986); Bai, Kim and Lee (1989a); Yum and Choi (1989); Seo and Yum (1991); Barton (1991); Bai and Chum (1991); Bai and Chung (1992); Bai and Kim (1993). Amongst them two main research works have been widely introduced giving main results with innovation. They are Nelson and Meeker (1978), Yum and Choi (1989).

Optimal Accelerated Life Test Designs problem was initially introduced by Chernoff (1962). He estimated the parameters describing the mean lifetime of a device under a standard environment. Five examples involving variations of models and experimental designs are studied. An exponential distribution of lifetimes is assumed in each of the five problems. The general approach, applying a method of Elfving, is applicable to a large variety of models.

Kielpinski and Nelson (1975) presents optimum accelerated life test plans for estimating a simple linear relationship between a stress and the median of product life which has a s-normal or lognormal distribution when the data are analyzed before all test units fail. Also, plans with equal numbers of test units at equally spaced test stresses are compared with the optimum plans. The plans are illustrated with a temperature-accelerated life test of electrical insulation.

Meeker and Nelson (1975) presents charts for optimum accelerated life test plans for estimating a simple linear relationship between an accelerating stress and product life, which has a Weibull or smallest extreme value distribution, when the data are to be analyzed before all test units fail. The plans show that one need not run all test units to failure and that more units ought to be tested at low test stresses than at high ones. The plans are illustrated with a voltage-accelerated life test of an electrical insulating fluid.

Nelson and Kielpinski (1976) presents theory for optimum plans for accelerated life tests for estimating a simple linear relationship between a stress and product life, which has as normal or lognormal distribution, when the data are to be analyzed before all test units fail. Standard plans with equal numbers of test units at equally spaced test stresses are presented and are compared with the optimum plans. While the optimum plans may not always be robust enough in practice, they indicate that more test units should be run at low stress than at high stress. The plans are illustrated with a temperature-accerlerated life test of an electrical insulation analyzed with the Arrhenius model.

Nelson and Meeker (1978) presents maximum likelihood theory for large sample optimum accelerated life test plans. The plans are used to estimate a simple linear relationship between (transformed) stress and product life, which has Weibull or smallest extreme value distribution. Censored data are to be analyzed before all test units fail. The plans show that all test units need not run to failure and that more units should be tested at low test stresses than at high ones. The plans are illustrated with a voltage-accelerated life test of an electrical insulating fluid.

DeGroot and Goel (1976) proposed a method of life testing which combines both ordinary and accelerated life testing produces. It is assumed that an item can be tested either in a standard environment or under stress. The amount of stress is fixed in advance and is the same for all items to be tested. However, the time x at which an item on test is taken out of the standard environment and put under stress can be chosen by the experimenter subject to a given cost structure. When an item is put under stress its lifetime is changed by the factor α. Let the random variable T denote the lifetime of an item in the standard environment, and let Y denote its lifetime under the partially accelerated test procedure just described. Then $y = T$ if $T \leq x$, and $Y = x + \alpha\ (T - x)$ if $T>x$. It is assumed that T has an exponential distribution with parameter θ. The estimation of θ and α and the optimal design of a partially accelerated life test are studied in the framework of Bayesian decision theory.

Miller and Nelson (1983) presents optimum plans for simple (two stresses) step-stress tests where all units are run to failure. Such plans minimize the asymptotic variance of the maximum likelihood estimator of the mean life at a design stress. The life-test model consists of: 1) an exponential life distribution with 2) a mean that is a log-linear function of stress, and 3) a cumulative exposure model for the effect of changing stress. Two types of simple step-stress tests are considered: 1) a time step test and 2) a failure step-test. A time-step test runs a specified time at the first stress, whereas, a failure-step test runs until a specified proportion of units fail at the first stress. New results include: 1) the optimum time at the first stress for time-step test and 2) the optimum proportion failing at the low stress for a failure-step test, and 3) the asymptotic variance of these optimum tests. Both the optimum time-step and failure-step tests have the same asymptotic variance as the corresponding optimum constant-stress test. Thus step-stress tests yield the same amount of information as constant-stress tests.

Meeker (1984) compares optimum test plans and some compromise test plans with respect to additional criterion including a) the ability to detect departures from the assumed stress life relationship and b) robustness to departures from the assumptions used in determining the plans. The comparisons are based on the large sample properties of maximum likelihood estimators, and the test plans are compared over a range of practical testing situations. The comparisons suggest some general rules for planning accelerated life tests.

Meeker (1986) gives guidelines for choosing statistically efficient inspection times and the approximate sample size that achieves a specified degree of precision for estimating a particular quantile of a Weibull time-to-failure distribution. This information can be used to plan more efficient lie tests.

Bai, Kim and Lee (1989a) presents the optimum simple time-step and failure-step stress accelerated life tests for the case where a prespecified censoring time is involved. An exponential life distribution with a mean that is a log-linear function of stress, and a cumulative exposure model are assumed. He obtained the optimum test plans to minimize the asymptotic variance of the maximum likelihood estimator of the mean life at a design stress. The optimum failure-step stress test plans are obtained in a closed form, whereas for time-step stress test nomograpahs are given for finding the optimum plans based on parameters that must be approximated from experience, similar data, or a preliminary test. For some selected values of the parameters, the effect of incorrect preestimates of the parameters in term of the percentage of variance increase has been studied and is small.

Yum and Choi (1989) determine optimal accelerated life test plans under the assumptions of periodic inspection and Type I censoring for the exponentially distributed lifetimes. Computational results indicate that for the range of parameter valuese considered the asymptotic variance of the estimated mean or p^{th} quantile at the use stress is not sensitive to the number of inspections at over stress levels. Sensitivity analyses are also conducted to see how sensitive the asymptotic variance of the estimated mean is with respect to the uncertainties involved in the guessed failure probabilities at the use and high stress levels. Computational results show that moderate deviations of the guessed failure probabilities from their true values are fairly tolerable in terms of the relative amount of increase in the asymptotic variance of the estimated mean. Procedures for selecting a sample size and for determining whether or not to conduct an accelerated life test are also discussed.

Seo and Yum (1991) develop statistically optimal and practical accelerated life test plans under the assumptions of intermittent inspection and Type-I censoring for the case where the lifetime4 at a stress level has a Weibull distribution. For a statistically optimal plan the low stress level, the proportion of test units allocated, and the inspection times are determined such that the asymptotic variance of the maximum-likelihood estimator of a certain quantile at the use condition is minimized. Although the practical plan adopts the same design criterion, it involves three rather than two overstress levels and easily calculated inspection schemes. Despite some loss in efficiency the practical plan has several advantages over the statistically optimal one. For instance, the practical plan can provide means for checking the validity of the assumptions made, may reduce the danger of extrapolation, and is more convenient to determine and implement. Computational experiments are conducted to evaluate the relative efficiency of a practical plan to the corresponding statistically optimal plan. Guidelines for selecting an appropriate practical plan are also described with an example.

Barton (1991) proposes a variation of the optimum accelerated life test plans described by Nelson and other, and shows how to minimize the maximum test-stress that is required,subject to meeting a certain standard-deviation limit on the estimate. Previous optimal life-test plans have shown how to minimize the standard-deviation of the estimated product life, subject to a given maximum test-stress.

Bai and Chun (1991) presents optimum simple step-stress accelerated life tests (ALTs) for products with competing causes of failure. The life distribution of each failure cause, which is independent of the others, is assumed to be exponential with a mean that is a log-linear function of the stress, and a cumulative exposure model is assumed. Optimum plans for time-step and failure-step ALTs are obtained which minimize the sum over all failure causes of asymptotic variances of the maximum likelihood estimators of the log mean lives at design stress. The competing causes of failure affect the optimum test plan only through the product of two ratios-the ratio of the sums of the mean lives and the ratio of the sums of the failure rates overall failure causes at low and high stress levels. The effect of this product (of two ratios) is studied.

Bai and Chung (1992) considers optimal designs for two partially accelerated life tests (PALTs) in which items are run at both accelerated and use conditions until a predetermined time. The step PALT allows the test to be changed from use to accelerated condition at a specified time; the constant PALT runs each item at either use or accelerated condition only. For items having constant hazard (failure) rate, maximum likelihood estimators (MLEs)of the hazard rate at use condition and the acceleration factor, the ratio of the hazard rate at accelerated condition to that at use condition, are obtained. The change-time for the step PALT or the sample proportion allocated to accelerated condition for the constant PALT is determined to minimize either the generalized asymptotic variance of MLEs of the acceleration factor and the hazard rate at use condition or the asymptotic variance of MLE of the acceleration factor.

Bai and Kim (1993) presents an optimum simple step-stress accelerated life test for the Weibull distribution under Type I censoring. It is assumed that a log-linear relationship exists between the Weibull scale parameter and the (possible transformed) stress and that a certain cumulative exposure model for the effect of changing stress hold. The optimum plan - low stress and stress change time - is obtained, which minimizes the asymptotic variance of the maximum likelihood estimator of a stated percentile at design stress. For selected values of the design parameters, nomographs useful for finding the optimumplan are given,and the effects of errors in preestimates of the parameters are investigated. As an alternative to the simple step-stress test, a three-level compromise plan is proposed, and its performance is studied and compared with that of the optimum simple step-stress test.

2.2 THEORY FOR OPTIMUM ACCELERATED CENSORED LIFE TESTS FOR WEIBULL AND EXTREME VALUE DISTRIBUTIONS

Nelson and Meeker (1978) considers ML estimation of a given percentile of a smallest extreme value (or Weibull) distribution at a design stress. Large-sample theory for optimum plans for simultaneous testing with Type I censored data is given.

Section 2.2.1 introduces the inverse power law model. Section 2.2.2 describes other aspects of the problem. Section 2.2.3 presents the optimum plans. Section 2.2.4 provides the theoretical basis of the plans.

2.2.1 THE MODEL

This section describes the inverse power law model, as it is used here with the Weibull distribution.

The model

The assumptions are:

1. Product life has a Weibull distribution at any stress. The

Weibull cumulative distribution function (cdf) is

$$G(t) = 1 - \exp[-(t/\alpha)^{\beta}], \quad t \geq 0 \; ; \qquad (2.2.1.1)$$

here $\alpha > 0$ and $\beta > 0$ are the Weibull scale and shape parameters, respectively.

2. The Weibull shape parameter β is constant, i.e., independent of stress.
3. The Weibull scale parameter α is an inverse power function of stress S. That is,

$$\alpha(S) = \exp(\gamma_0)/S^{\gamma_1} \qquad (2.2.1.2)$$

4. The lifetimes of the test units are independent of each other.

Equation (2.2.1.2) is called the inverse power law. The parameters γ_0, γ_1, and β are characteristic of the product ; they are to be estimated from the data.

If product life T has a Weibull distribution, the natural logarithm of life, $y = \ln(T)$ has a smallest extreme value distribution. Its cdf is

$$F(y) = 1 - \{\exp[-\exp[(y - \mu)/\sigma]]\} \;, \quad \infty < y < \infty \; ; \qquad (2.2.1.3)$$

here $\sigma > 0, -\infty < \mu < \infty$, $\sigma = 1/\beta$, and $\mu = \ln(\alpha)$,
Then, (2.2.1.2) can be written as

$$\mu(S) = \gamma_0 + \gamma_1 \ln(1/S) \qquad (2.2.1.4)$$

It is convenient to define a transformed stress, $x = \ln(1/S)$; it will be referred to as stress.

2.2.2 THE OPTIMIZATION PROBLEM

This section describes the optimization criterion, the assumed testing method, constraints, and other necessary background material.

The optimization criterion

The ML elimination of the 100P-th percentile of the smallest, extreme value distribution at stress x is

$$\hat{y}_p(x) = \hat{\gamma}_0 + \hat{\gamma}_1 x + u_p \hat{\sigma} \; ; \tag{2.2.2.1}$$

here $u_p = \ln[-\ln(1-P)]$. The corresponding ML estimate of the Weibull percentile is

$$\hat{t}_p(x) = \exp(\hat{y}_p(x)) \; . \tag{2.2.2.2}$$

The optimization criterion used here is to minimize the large sample variance of (2.2.2.1) at a specified (usually design) stress. This is equivalent to minimizing the large sample variance of $\hat{t}_p(x)/t_p(x)$, that is, minimizing the relative (percentage) error.

The optimum test plan specifies the two optimum test stresses and the proportion of test units allocated to each.

Constraints

It is assumed that (I) all test units start on test at same time and are run simultaneously, (2) failed units are not replaced, and (3) the test is terminated at a prechosen censoring time η. Schedule and cost constraints usually determine η. The high test stress X_H should be as high as possible: of course, the linear model (2.2.1.4) must hold from x_S to x_H. Maximizing x_H minimizes the large sample variance of $\hat{y}_p(x_S)$.

Assumed parameter values

The optimum plan depends on the true model parameter values. In practice one must approximate the parameters, using experience, similar data, or a preliminary test.

Assuming a smallest extreme value distribution suppose that the distribution at the specified highest test stress X_H has a location parameter $\mu_H = \gamma_0 + \gamma_1 x_H$. The standardized censoring time is

$$a = (\eta - \mu_H)/\sigma \; ; \tag{2.2.2.3}$$

here η is the actual censoring time. Similarly, the standardized slope is

$$b = \gamma_1(X_S - X_H)/\sigma = (\mu_S - \mu_H)/\sigma \tag{2.2.2.4}$$

The quantities a and b completely characterize the problem and must be estimated.

2.2.3 OPTIMUM PLANS

This section describes optimum test plans that minimize the (relative) variance of the maximum likelihood estimate of a given extreme value (Weibull) percentile at the specified design stress X_S. Optimum test plans for this problem use only two test stresses.

The optimum test stresses

The optimum low test stress is

$$x_L = x_H + \xi(x_S - x_H) \quad . \tag{2.2.3.1}$$

Here the optimum value for ξ is a function of a and b and the percentage 100P of the given percentile; ξ is obtained as shown in Section 2.2.4. Also, Meeker and Nelson (1975) give a chart for the optimum ξ as a function of the standardized values a and b and of P.

The optimum allocation

The optimum proportion π of test units to run at the optimum low test stress is obtained with the method described in Section 2.2.4 Meeker and Nelson (1975) provide charts for the optimum π as a function of a,b, and the percentage 100P of the given percentile. The charts show that, in practical applications, the majority of the test units should be run at the test stress x_L near the design stress.

Variance of the estimate

For an optimum plan with a sample size n, the variance of $\hat{y}_P(x_S)$ is:

$$\mathrm{Var}[\hat{y}_P(Xs)] = \sigma^2 V/n; \qquad (2.2.3.2)$$

here the variance factor V is a function of a, b and P.

The variance of the corresponding estimate of the Weibull percentile is

$$\mathrm{Var}(\hat{t}_P) = t_P^2 V/(n\beta^2) \qquad (2.2.3.3)$$

where V is the same as in (2.2.3.2).

2.2.4 THEORY

This section presents maximum likelihood theory for estimation of the model and the expression for the variance of the estimate of a given percentile at a design stress. The variance to be minimized is a function of the model parameters a and b,the test stresses, the allocation of the test units to each stress, and P.

Reparametrized model

Define the stress factor

$$\xi_1 = (x - x_H)/(x_S - x_H) \quad . \qquad (2.2.4.1)$$

As before, x_H is the highest test stress and is specified; x_S is the design stress where the given percentile is to be estimated. For the highest test stress $x = x_H$, $\xi_1 = 0$; and , for the design stress $x = x_S$, $\xi_1 = 1$. The relationship (2.2.1.2) for the location parameter of the smallest extreme value distribution may be written in terms of ξ_1 as

$$\mu(\xi_1) = \beta_0 + \beta_1\xi_1 \;; \tag{2.2.4.2}$$

here new coefficients β_0 and β_1 are related to the previous γ_0 and γ_1 by

$$\beta_0 = \gamma_0 + \gamma_1 x_H = \mu_H \tag{2.2.4.3}$$

and

$$\beta_1 = \gamma_1(x_S - x_H) = \mu_S - \mu_H \;. \tag{2.2.4.4}$$

As before, μ_S and μ_H are the location parameters at x_S and x_H, respectively. The 100Pth percentile $y_P(1) = \beta_0 + \beta_1\xi_1 + u(P)\sigma$ is to be estimated where $u(P) = \ln[-\ln(1-P)]$. The scale parameter σ is the same for both forms of the model. To simplify the derivation, (2.2.4.2) is written as

$$\mu(\xi_1) = \beta_0\xi_0 + \beta_1\xi_1 \tag{2.2.4.5}$$

where $\xi_0 = 1$.

Log likelihood

First define the indicator function $I = I(y)$ in terms of the censoring time η by

$$I = \begin{cases} 1 \text{ if } y \le \eta, \text{ failure before time } \eta \;, \\ 0 \text{ if } y > \eta, \text{ censored at time } \eta \;. \end{cases} \tag{2.2.4.6}$$

Let

$$z = (y-\mu(\xi_1))/\sigma = (y-\beta_0\xi_0-\beta_1\xi_1)/\sigma \tag{2.2.4.7}$$

be a standardized failure time, and let

$$\zeta = (\eta-\mu(\xi_1))/\sigma = (\eta-\beta_0\xi_0-\beta_1\xi_1)/\sigma \qquad (2.2.4.8)$$

be a standardized censoring time. Also, let

$$\phi = \phi(\zeta) = \phi((\eta-\beta_0\xi_0-\beta_1\xi_1)/\sigma) \qquad (2.2.4.9)$$

where $\phi(\zeta)$ is the cumulative distribution function of the standard smallest extreme value distribution, i.e. for $\mu = 0$ and $\sigma = 1$.

The log likelihood L of a Type I censored observation at a stress ξ_1 is

$$L = I[-\ln(\sigma)-e^z+z) + (1-I)\ln(1-\phi) \ . \qquad (2.2.4.10)$$

Suppose the i^{th} observation y_i corresponds to a value ξ_{1i} and the corresponding log likelihood is L_i. Then the sample log likelihood L_0 for n independent observation is

$$L_0 = L_1 + \ldots + L_n. \qquad (2.2.4.11)$$

The maximum likelihood estimates $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\sigma}$ are the parameters values that maximize the sample log likelihood (2.2.4.11)

Fisher information matrix

From(2.2.4.10), the Fisher information matrix F_{ξ_1} is obtained by taking the negative expectations of the second partial and mixed partial derivatives of L with respect to the model parameters, namely.

$$E\{-\partial^2L/\partial\beta_j\partial\beta_k\} = (\xi_j\xi_k/\sigma^2)\{\phi\}, \ j,k = 0,1 \qquad (2.2.4.12)$$

$$E\{-\partial^2 L/\partial\beta_j\partial\sigma\} = \xi_j/\sigma^2)\left\{ \int_0^{e^\zeta}[\ln(w)]we^{-w}dw + (1-\phi)\zeta\ e^\zeta \right\},\ j = 0,1$$

$$E\{-\partial^2 L/\partial\sigma^2\} = (1/\sigma^2)\left\{ \phi + \int_0^{e^\zeta} [\ln(w)]^2 we^{-w}\ dw + (1-\phi\zeta^2 e^\zeta \right\}.$$

Since ϕ is a function of just ζ, the last three expressions in braces {} are functions of ζ. Denote them by $A(\zeta)$, $B(\zeta)$, and $C(\zeta)$, respectively. For $\xi_0 = 1$, the Fisher information matrix F_{ξ_1} for an observation at ξ_1 has the form

$$F_{\xi_1} = \frac{1}{\sigma^2}\begin{bmatrix} A(\zeta) & \xi_1 A(\zeta) & B(\zeta) \\ \xi_1 A(\zeta) & \xi_1{}^2 A(\zeta) & \xi_1 B(\zeta) \\ B(\zeta) & \xi_1 B(\zeta) & C(\zeta) \end{bmatrix} \qquad (2.2.4.13)$$

The quantity ζ can be written in terms of the standardized quantities a and b as

$$\zeta = [(\eta-\mu_H)/\sigma] - [(\mu_S-\mu_H)/\sigma]\xi_1 = a-b\xi_1\ . \qquad (2.2.4.14)$$

The Fisher information matrix for any plan with a sample of n independent observations is the sum of the matrices $F_{\xi_{1i}}$ for each individual observation:

$$F = \sum_{i=1}^{n} F_{\xi_{1i}} \qquad (2.2.4.15)$$

where the i^{th} unit is tested at a ξ_1 value of ξ_{1i}. The Fisher information matrix for test of n units is

$$F = n(1-\pi)F_0 + n\pi F_\xi = (n/\sigma^2)(1-\pi)\begin{bmatrix} A(a) & 0 & B(a) \\ 0 & 0 & 0 \\ B(a) & 0 & C(a) \end{bmatrix}$$

$$+ (n/\sigma^2)\pi \begin{bmatrix} A(a-b\xi) & \xi A(a-b\xi) & B(a-b\xi) \\ \xi A(a-b\xi) & \xi^2 A(a-b\xi) & \xi B(a-b\xi) \\ B(a-b\xi) & \xi B(a-b\xi) & C(a-b\xi) \end{bmatrix} \qquad (2.2.4.16)$$

$(\sigma^2/n)F$ is a function of a,b,ξ, and π.

Variance of the estimate of a design percentile

For any plan, the asymptotic covariance matrix V of the maximum likelihood estimates β_0, β_1, and $\hat{\sigma}$ is the inverse of the corresponding Fisher information matrix. That is,

$$V = \begin{bmatrix} \mathrm{Var}(\hat{\beta}_0) & \mathrm{Cov}(\hat{\beta}_0,\hat{\beta}_1) & \mathrm{Cov}(\hat{\beta}_0,\hat{\sigma}) \\ \mathrm{Cov}(\hat{\beta}_0,\hat{\beta}_1) & \mathrm{Var}(\hat{\beta}_1) & \mathrm{Cov}(\hat{\beta}_1,\hat{\sigma}) \\ \mathrm{Cov}(\hat{\beta}_0,\hat{\sigma}) & \mathrm{Cov}(\hat{\beta}_1,\hat{\sigma}) & \mathrm{Var}(\hat{\sigma}) \end{bmatrix} = F^{-1} \qquad (2.2.4.17)$$

The maximum likelihood estimate of the 100P-th percentile of the distribution at the design stress x_S is

$$\hat{y}_P(1) = \hat{\beta}_0 + \beta_1.1 + u_P\hat{\sigma} \qquad (2.2.4.18)$$

where $u_P = \ln[-\ln(1-P)]$ is the $100P^{th}$ percentile of the standard smallest extreme value distribution. The corresponding asymptotic variance of the estimate is the value of the quadratic form

$$\mathrm{Var}(\hat{y}_P(1)) = \begin{bmatrix} 1 & 1 & u_P \end{bmatrix} V \begin{bmatrix} 1 \\ 1 \\ u_P \end{bmatrix} \qquad (2.2.4.19)$$

Optimum Plans

For a plan with two stresses and unequal allocation, the variance (2.2.4.19) is a function of a, b, P, ξ, and π. Then ξ and π can be chosen to minimize this variance for given values of a, b, and P.

2.3 OPTIMAL DESIGN OF ACCELERATED LIFE TESTS UNDER PERIODIC INSPECTION

Yum and Choi (1989) developed an asymptotically optimal ALT plan for the exponentially distributed lifetimes under the assumptions of Type I censoring and periodic inspection. The optimally criterion adopted is the minimum variance of the estimated mean or pth quantile of the lifetime distribution at the use stress. Maximum likelihood (ML) estimation method is used to estimate the unknown parameters in the relationship between the mean lifetime and the stress. Computational experiments are conducted to examine how optimal plans vary with respect to the parameters involved. Sensitivity analyses are also performed to assess the effect of inaccuracy in the "guesstimates" of the unknown parameters on the optimal plan.

2.3.1 THE MODEL

Assume that the lifetimes (τ) of test items are independently and identically distributed as exponential. That is

$$f(t) = (1/\theta)e^{(-t/\theta)}, \qquad t,\theta > 0. \qquad (2.3.1.1)$$

The mean lifetime θ and the stress s are assumed to be related as

$$\theta = e^{(\beta_0+\beta_1 s)} \qquad (2.3.1.2)$$

Three stress levels are considered. That is, the use stress level s_0, the low stress level s_1, and the high stress level s_2. The number of test items allocated to s_1 and s_2 are respectively given by

$$n_1 = \alpha_1 N, \qquad n_2 = \alpha_2 N = (1-\alpha_1)N. \qquad (2.3.1.3)$$

At s_i, n_i units are to be put on test at time 0 and run until a prespecified time t_{ci}, and inspections are conducted only at specified points in time t_{i1}, $t_{i2}, \ldots, t_{i,K(i)}$ where $t_{i,K(i)}=t_{ci}$. Let $t_{i0}=0$ and $t_{i,K(i)+1}=\infty$, and define

$$x_{ij} = \text{the number of items failed in } (t_{i,j-1}, t_{ij}),$$
$$P_{ij} = \text{the probability of failure in } (t_{i,j-1}, t_{ij}),$$
$$j = 1,2,\ldots,K(i)+1.$$

The grouped data $\{x_{ij},\ i=1,2;\ j=1,2,\ldots,K(i)+1\}$ are used to estimate β_0 and β_1 in Eq. (2.3.1.2). The estimated relationship is then extrapolated to estimate some quantities at the use condition. The logarithm of the mean lifetime at the use condition is defined by

$$\mu_0 = \ln\theta_0 = \beta_0 + \beta_1 s_0. \tag{2.3.1.4}$$

Note that t_p, the pth quantile of the exponential distribution at the use condition, is related to μ_0 as follows.

$$y_p = \ln t_p = \mu_0 + \ln\{-\ln(1-p)\}. \tag{2.3.1.5}$$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be some estimators of β_0 and β_1, respectively. Then,

$$\hat{y}_p = \hat{\mu}_0 + \ln\{-\ln(1-p)\}. \tag{2.3.1.6}$$

The problem of optimally designing the ALT plan under periodic inspection can now be stated as given N, s_0, s_2, $\{t_{ij},\ i=1,2;\ j=1,2,\ldots K(i)\}$ determine α_1 and s_1 such that the variance of $\hat{\mu}_0$ (or, equivalently the variance of $\hat{y}_p$) is minimized.

2.3.2. MAXIMUM LIKELIHOOD ESTIMATION AND OPTIMAL PLANS

At s_i, the grouped data $\{x_{ij},\ j = 1,2,\ldots,K(i)+1\}$ are multinomially distributed with parameters n_i and $\{P_{ij},\ j =$

$1,2,\ldots,K(i)+1\}$. The Log Likelihood function is given by

$$L = C + \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} x_{ij} \ln P_{ij} \tag{2.3.2.1}$$

where C is a constant and

$$P_{ij} = e^{-(t_{i,j-1}/\theta_i)} - e^{-(t_{ij}/\theta_i)}$$

Then, the ML estimates of β_0 and β_1 are obtained by solving the following equations.

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} \left(x_{ij}(B_{i,j-1} - B_{ij})/P_{ij}\right) = 0 \tag{2.3.2.2}$$

$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{2} s_i \sum_{j=1}^{K(i)+1} \left(x_{ij}(B_{i,j-1} - B_{ij})/P_{ij}\right) = 0 \tag{2.3.2.3}$$

where $B_{ij} = \left(t_{ij}/\theta_i\right) e^{-\left(t_{ij}/\theta_i\right)}$

The **Fisher information matrix** is given by

$$F = N\left(f_{gh}\right); \qquad g,h = 0,1 \tag{2.3.2.4}$$

$$f_{gh} = \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{K(i)+1} \left(\frac{\partial P_{ij}}{\partial \beta_g}\right)\left(\frac{\partial P_{ij}}{\partial \beta_h}\right) / P_{ij}$$

The asymptotic variance (avar) of $\hat{\mu}_0$ is given by

$$\text{avar}(\hat{\mu}_0) = N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}(f_{11} + s_0^2 f_{00} - 2s_0 f_{01}) \tag{2.3.2.5}$$

The optimization problem is to determine s_1 and α_1 that minimize avar$(\hat{\mu}_0)$. Set $s_0=0$. Then (2.3.2.5) is reduced to

$$\text{avar}(\hat{\mu}_0) = N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}f_{11} .$$

After some algebraic manipulation we obtain

$$\text{avar}(\hat{\mu}_0) = N^{-1}(s_2^2Q_2 + (s_1^2Q_1 - s_2^2Q_2)\alpha_1)/ (Q_1Q_2(s_1-s_2)^2(-\alpha_1^2+\alpha_1)) \tag{2.3.2.6}$$

where

$$Q_i = \sum_{j=1}^{K(i)+1} \left(B_{i,j-1} - B_{ij}\right)^2/P_{ij} \tag{2.3.2.7}$$

The following two-step procedure is adopted to minimize avar$(\hat{\mu}_0)$ with respect to s_1 and α_1.

First, optimize α_1 for given s_2. The optimal value α_1^* of α_1, is given by

$$\alpha_1^* = \left(-s_2^2Q_2+\sqrt{s_1^2s_2^2Q_1Q_2}\right)/\left(s_1^2Q_1-s_2^2Q_2\right) \tag{2.3.2.8}$$

Second, α_1^* and the minimum of avar$(\hat{\mu}_0)$ are obtained on the grid s_1 = d, 2d, 3d, ..., where d is the grid size.

When K(1) and K(2) go to infinity

$$Q_i = 1 - e^{(-t_{ci}/\theta_i)}. \tag{2.3.2.9}$$

2.3.3 COMPUTATIONAL RESULTS

For the purpose of computational experiments, set $t_{C1}=t_{C2}=t_C'$ and K(1)=K(2)=K. Furthermore, parameters are standardized such that the high stress levels as well as the common censoring time becomes 1. Such standardization does not alter the nature of the problem. In actual experiments the following quantities are used instead of β_0 and β_1. At the used condition

P_u = probability that an item fails in (0,1) and

P = probability that an item fails in (0,1) at the high stress.

Then, the corresponding β_0 and β_1 can be determined as follows.

$$\beta_0 = \ln\{-1/\ln(1-P_u)\}, \tag{2.3.3.1}$$

$$\beta_1 = \ln\{\ln(1-P_u)/\ln(1-P_h)\} \tag{2.3.3.2}$$

Besides, the grid size for s_1 is set to 0.002. Finally, N is set to 1, since it only serves as a scale factor for avar$(\hat{\mu}_0)$.

Computational results show that avar$(\hat{\mu}_0)$ is not sensitive to K. For all the cases considered, avar$(\hat{\mu}_0)$ for K=1,2, or 3 is sufficiently close to avar$(\hat{\mu}_0)$ for $K = \infty$. This implies that designing an ALT under periodic inspection the number of K needs not be too large. Also, the optimal low stress level (s_1^*) and the optimal proportion of test items allocated to the low stress (α_1^*) are rather stable over K.

Finally, as P_u increases and/or P_h decreases s_1^* gets closer to zero and α_1^* to one.

CHAPTER 3

OPTIMAL DESIGN OF ACCELERATED LIFE TEST PLANS UNDER PERIODIC INSPECTION AND TYPE I CENSORING: THE CASE OF RAYLEIGH FAILURE LAW

3.1 INTRODUCTION

Some products, materials and electronic devices has high reliability. Testing such devices under usual operating conditions is difficult, because the devices are not likely to fail in the available time-period for tests. Accelerated life tests(ALT) quickly provide information on life distributions of products and materials, which saves time as well as cost over testing, further time and cost are reduced by employing periodic inspection in which test items are checked only at certain points in time. Yum and Choi (1989) attempted to combine acceleration and periodic inspection and developed an asymptotically optimal ALT plan for the exponentially distributed life times under the assumptions of Type I censoring and periodic inspection.

The present investigation is an attempt to combine such interesting and important features of life tests as acceleration and periodic inspection. The Rayleigh failure distribution is considered to describe the life times of test units. An asymptotically optimal ALT plan is developed under Type-I censoring and periodic inspection at two stress levels. Inspection times at each stress level are equally spaced between initial and censored times. Maximum likelihood (ML) estimation method is used to estimate the parameters involved in the relationship between mean life time and the stress. A self developed software has been used to carry out the compuvations of asymptotic variance of the ML estimator. The asymptotic variance of the estimated mean or p^{th} quantile of the life time distribution at the use condition is adopted as a criterion for determining optimal design. Sensitivity analyses has been carried out to assess the effect of inaccuracy in the 'guesstimates' of the unknown parameters on the optimum plan.

3.2 THE MODEL

Let us assume that the life times (T) of test items at stress level s_i are independently and identically distributed as Rayleigh with the probability density function

$$f(t) = \frac{t}{\theta^2} e^{(-t^2/2\theta^2)} \quad ; \; t, \; \theta > 0 \; . \qquad (3.2.1)$$

Assume that the mean life time θ and the stress s are related as (i.e see Miller and Nelson(1983), Nelson(1990)):

$$\ln\theta = \beta_0 + \beta_1 S \; ,$$
$$\theta = e^{(\beta_0 + \beta_1 s)} \; , \qquad (3.2.2)$$

where β_0 and β_1 are parameters of the product and the test method and we have to estimate these parameter. This relationship is frequently used in ALT. In fact, it can be shown that if s is the log of voltage stress , then (3.2.2) is the inverse power law and if s is the reciprocal of absolute temperature, then (3.2.2) is the Arrhenius relation.

We are considering three stress levels s_0, s_1 and s_2. s_0 is the use stress level, the low stress level is s_1 and the high stress level is s_2. s_0 and s_2 are prespecified, and $s_1 (s_1 < s_2)$ is to be optimally determined. The number of test items allocated to s_1 and s_2 are given by

$$n_1 = \alpha_1 N, \quad \alpha_1 + \alpha_2 = 1 \; , \qquad (3.2.3)$$

$$n_2 = \alpha_2 N = (1 - \alpha_1) N , \tag{3.2.4}$$

where N is the total number of test items given and α_1 is to be optimally determined.

At s_i, n_i units are to be put on test at time 0 and run until a pre-specified time t_{ci} (i.e. Type I censoring is assumed), and inspections are conducted only at specified points in time $t_{i1}, t_{i2}, \ldots, t_{i,K(i)}$, where $t_{i,K(i)} = t_{ci}$.

In addition, let $t_{io} = 0$ and $t_{i,K(i)+1} = \infty$, and define

$$x_{ij} = \text{the number of items (at stress level } s_i\text{) failed in } (t_{i,j-1}, t_{ij}) , \quad \text{for } j = 1,2,\ldots,K(i)+1; \tag{3.2.5}$$

$$P_{ij} = \text{the probability of failure at stress level } s_i \text{ in } (t_{i,j-1}, t_{ij}) \quad \text{for } j = 1,2,\ldots,K(i)+1; \tag{3.2.6}$$

Clearly,

$$P_{ij}(t_{i,j-1}, t_{ij};\theta) = \int_{t_{i,j-1}}^{t_{ij}} f(t;\theta)dt = F(t_{ij};\theta) - F(t_{ij-1};\theta)$$
$$= e^{-t_{i,j-1}^2/2\theta^2} - e^{-t_{ij}^2/2\theta^2} .$$

Also $F(t_{io};\theta) = 0$ and $F(t_{i,K(i)+1};\theta) = 1$.

Then, the resulting structure of periodic inspection at s_i can be described as follows:

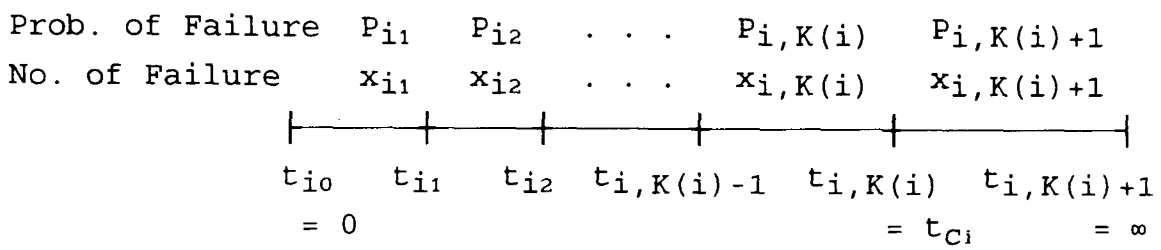

Figure 1. Structure of periodic inspection at the i^{th} stress level.

Consider the grouped data $\{x_{ij},\ i = 1, 2;\ j = 1, 2, \ldots K(i)+1\}$ which are used to estimate β_0 and β_1 in equation (3.2.2), and then the estimated relationship is extrapolated to estimate the mean life time at the use condition. The logarithm of mean lifetime at the use condition is given by

$$\mu_0 = \ln \theta_0 = \beta_0 + \beta_1 s_0 \tag{3.2.7}$$

We are interested in estimating the p^{th} quantile (t_p) of the Rayleigh distribution at the use condition.

The p^{th} quantile (t_p) of the life distribution at a stress level s_0 is:

$$t_p = \sqrt{2}\theta_0\{-\ln(1-P)\}^{1/2}.$$

Thus,

$$\begin{aligned} \ln t_p &= \ln\sqrt{2} + \ln\theta_0 + 1/2 \ln[-\ln(1-P)] \\ &= 1/2 \ln 2 + \mu_0 + 1/2 \ln\{-\ln(1-p)\} \\ &= 1/2 \ln 2 + \beta_0 + \beta_1 s_0 + 1/2 \ln[-\ln(1-P)] \\ &= y_p \text{ (say)} . \end{aligned} \tag{3.2.8}$$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ are some estimators of β_0 and β_1, respectively. Then, y_p is estimated as,

$$\hat{y}_p = 1/2 \ln 2 + \hat{\beta}_0 + \hat{\beta}_1 s_0 + 1/2 \ln\{-\ln(1-p)\} \tag{3.2.9}$$

3.3. MAXIMUM LIKELIHOOD ESTIMATION

The grouped data $\{x_{ij},\ j = 1,2,\ldots,K(i)+1\}$ are multinomially distributed with parameters n_i and $\{P_{ij},\ j = 1,2,\ldots,K(i)+1\}$, at stress level s_i. The likelihood function is given by

$$L' = \prod_{i=1}^{2} L_i'$$

$$= \prod_{i=1}^{2} n_i! \left[\prod_{j=1}^{K(i)+1} x_{ij}! \right]^{-1} \cdot \left[\prod_{j=1}^{K(i)+1} P_{ij}{}^{x_{ij}} \right] . \qquad (3.3.1)$$

Taking logarithm of both sides

$$L = \ln L' = \sum_{i=1}^{2} \ln L_i'$$

$$= C + \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} x_{ij} \ln P_{ij} , \qquad (3.3.2)$$

where C is a constant and P_{ij} is given by

$$P_{ij} = e^{-\left(t_{i,j-1}/\sqrt{2}\theta_i\right)^2} - e^{-\left(t_{ij}/\sqrt{2}\theta_i\right)^2}, \qquad (3.3.3)$$

for i = 1, 2 and j = 1,2, ..., K(i)+1,

Then, the M L estimates β_0 and β_1 are values of β_0 and β_1 which solve the equations obtained by letting the first partial derivatives of L with respect to β_0 and β_1 to be zero.

That is,

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} \left(x_{ij} (B_{i,j-1} - B_{ij}) / P_{ij} \right) = 0 . \qquad (3.3.4)$$

$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{2} s_i \sum_{j=1}^{K(i)+1} \left(x_{ij} (B_{i,j-1} - B_{ij}) / P_{ij} \right) = 0 . \qquad (3.3.5)$$

Note that, for i = 1, 2 and j = 0,2, ..., K(i) + 1,

$$B_{ij} = (t_{ij}/\theta_i)^2 \, e^{-(t_{ij}/\sqrt{2}\theta_i)^2} . \qquad (3.3.6)$$

These estimates can also be found out by Monte Carlo studies.

3.4 ASYMPTOTIC COVARIANCE OF THE M L ESTIMATOR

The Fisher information matrix F is obtained by taking negative s-expectations of second partial derivatives of L with respect to β_0 and β_1. That is,

$$F = \begin{bmatrix} \sum_{i=1}^{2} E\left\{-\dfrac{\partial^2 L}{\partial \beta_0^2}\right\} & \sum_{i=1}^{2} E\left\{-\dfrac{\partial^2 L}{\partial \beta_0 \partial \beta_1}\right\} \\ \sum_{i=1}^{2} E\left\{-\dfrac{\partial^2 L}{\partial \beta_1 \partial \beta_0}\right\} & \sum_{i=1}^{2} E\left\{-\dfrac{\partial^2 L}{\partial \beta_1^2}\right\} \end{bmatrix}$$

$$= N(f_{gh}) ; \; g, \; h = 0, \; 1 \qquad (3.4.1)$$

$$f_{gh} = \sum_{i=1}^{2} a_i \sum_{j=1}^{K(i)+1} \left(\frac{\partial P_{ij}}{\partial \beta_g}\right)\left(\frac{\partial P_{ij}}{\partial \beta_h}\right) / P_{ij} . \qquad (3.4.2)$$

After some algebraic manipulation, we obtain

$$f_{00} = \sum_{i=1}^{2} a_i \sum_{j=1}^{K(i)+1} (B_{i,j-1} - B_{ij})^2 / P_{ij} . \qquad (3.4.3)$$

$$f_{01} = f_{10} = \sum_{i=1}^{2} a_i s_i \sum_{j=1}^{K(i)+1} (B_{i,j-1} - B_{ij})^2 / P_{ij} . \qquad (3.4.4)$$

$$f_{11} = \sum_{i=1}^{2} \alpha_i s_i^2 \sum_{j=1}^{K(i)+1} \left(B_{i,j-1} - B_{ij}\right)^2/P_{ij} \; . \qquad (3.4.5)$$

where

$$B_{ij} = \left(t_{ij}/\theta_i\right)^2 e^{-\left(t_{ij}/\sqrt{2}\theta_i\right)^2} ,$$

for $i = 1,2$ and $j = 0,1,2,\ldots K(i)+1$.

The asymptotic covariance matrix of the maximum likelihood estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ is the inverse of the Fisher information matrix. That is,

$$F^{-1} = N^{-1}\begin{bmatrix} \mathrm{var}(\hat{\beta}_0) & \mathrm{cov}(\hat{\beta}_0,\hat{\beta}_1) \\ \mathrm{cov}(\hat{\beta}_1,\hat{\beta}_0) & \mathrm{var}(\hat{\beta}_1) \end{bmatrix}$$

$$= N^{-1}\left(f_{00}f_{11} - f_{01}^2\right)^{-1}\begin{pmatrix} f_{11} & -f_{01} \\ -f_{10} & f_{00} \end{pmatrix} . \qquad (3.4.6)$$

From eqn. (3.2.9), the corresponding asymptotic variance (avar) of $\hat{y}_p$ is obtained as

$$\mathrm{avar}(\hat{y}_p) = \mathrm{avar}(\hat{\beta}_0 + \hat{\beta}_1 s_0)$$

$$= (1,\ s_0)F^{-1}\binom{1}{s_0}$$

$$= N^{-1}\left(f_{00}f_{11} - f_{01}^2\right)^{-1}\ (1,\ s_0)\begin{pmatrix} f_{11} & -f_{01} \\ -f_{10} & f_{00} \end{pmatrix}\binom{1}{s_0}$$

$$= N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}(f_{11} + s_0^2 f_{00} - 2s_0 f_{01}) \; . \qquad (3.4.7)$$

The problem of optimally designing the ALT plan under periodic inspection can now be stated as: given N, s_0, s_2, $\{t_{ci},\ i = 1,2\}$ determine α_1, s_1 and $\{t_{ij},\ i = 1,2;\ j = 1,2,\ldots,K(i)\}$ such that $\mathrm{avar}(\hat{y}_p)$ is minimized.

3.5 STATISTICALLY OPTIMAL PLANS

We consider the following assumptions:

(1) Censoring times at s_1 and s_2 are the same.
That is, $t_{C1} = t_{C2} = t_C$.

(2) The number of inspection at each stress level is the same.
That is, $K(1) = K(2) = K$.

Furthermore, parameters are standardized such that common censoring time (t_C) is 1, $s_0 = 0$, and $s_2 = 1$. The nature of the problem is remain same under the above standardization (see Appendix A).

Under the above assumptions, optimal plans are developed by determining optimal values of s_1 and α_1 for given N and K such that $\text{avar}(\hat{y}_p)$ is minimized. The optimization procedure is initiated by first 'guess estimate' of P_u and P_h which are more familiar and easier to estimate than β_0 and β_1. That is,

$P_u = P_r\{$an item fails in $(0, t_C = 1)$ at the use condition $\}$

$P_h = P_r\{$an item fails in $(0, t_C = 1)$ at the high stress$\}$

Then, β_0 and β_1 are determined as follows:

$$\beta_0 = \frac{1}{2} \ln\{-1/2\ln(1 - P_u)\} \tag{3.5.1}$$

$$\beta_1 = \frac{1}{2} \ln\{\ln(1 - P_u)/\ln(1 - P_h)\} . \tag{3.5.2}$$

Based upon the above standardization the equation (3.4.7) becomes

$$\text{avar}(\hat{y}_p) = N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}f_{11} . \tag{3.5.3}$$

After some algebraic manipulation we obtain

$$\text{avar}(\hat{y}_p) = N^{-1}(Q_2 + (s_1^2 Q_1 - Q_2)\alpha_1)/(Q_1 Q_2 (s_1 - 1)^2 \cdot (-\alpha_1^2 + \alpha_1)) , \tag{3.5.4}$$

where

$$Q_i = \sum_{j=1}^{K(i)+1} (B_{i,j-1} - B_{ij})^2/P_{ij} \tag{3.5.5}$$

for i = 1, 2.

For given P_u and P_h, the minimum value of $\text{avar}(\hat{y}_p)$ with respect to s_1 and α_1 is determined by two-step procedure, which was developed by Yum and Choi (1989). A detailed description of two-step procedure is given in Appendix B.

When K(1) = K(2) = K go to infinity (i.e. continuous inspection) then minimum $\text{avar}(\hat{y}_p)$ and corresponding optimal s_1 (i.e., s_1^*) and α_1 (i.e., α_1^*) were determined by using the method described in Nelson and Meeker's (1978) paper. It can be also determined by two-step procedure discussed in Appendix B. Finally, ratio of $\text{avar}(\hat{y}_p(K))$ to $\text{avar}(\hat{y}_p(\infty))$ was determined.

3.6 RESULTS AND DISCUSSIONS

The results are summarized in Tables 1-8. Tables 1-4, show statistically optimal plans whereas Tables 5-8, show sensitivity analysis of $\text{avar}(\hat{y}_p)$, for various combinations of P_u, P_h and K. We observe that $\text{avar}(\hat{y}_p)$ is not sensitive to K. Also, s_1^* and α_1^* are fairly stable over K for given P_u and P_h. It is clear that $\text{avar}(\hat{y}_p)$ for K = 2, 5 or 10 is sufficiently close to that for $K = \infty$. This implies that the number of inspections (K) need not be too large. This is an achievement in life testing of an item which has high reliability. Ratio of $\text{avar}(\hat{y}_p(K))$ to $\text{avar}(\hat{y}_p(\infty))$ is taken in last column of tables, which is approximately unity. This implies that there is no significant difference in asymptotic variance of $\hat{y}_p$, which is an encouraging result is terms of testing efforts. Also, when P_u increases and P_h decreases, the value of s_1^* is close to the use stress and the value of α_1^* tends to 1. That is, in particular, when $P_u = 0.1$, $s_1^* \cong 0$ and $\alpha_1^* \cong 1$ for $P_h \le 0.9$. This shows that there is almost no need for an ALT. Similar trends are observed when the values of P_h are very small for all P_u.

Sensitivity analysis, is conducted with respect to plausible values of P_u and P_h as $\tilde{P}_u$ and $\tilde{P}_h$. We first determine optimal s_1 and α_1 denoted as $\tilde{s}_1^*$ and $\tilde{\alpha}_1^*$ for $\tilde{P}_u$ and $\tilde{P}_h$ and then the ratio, $\text{avar}(\hat{y}_p(\tilde{s}_1^*, \tilde{\alpha}_1^*))/\text{avar}(\hat{y}_p(s_1^*, \alpha_1^*))$ is calculated for various cases with $K = 2$. These are shown in Tables 5-8. We observe that ratios are very close to 1 in general. This implies that $\text{avar}(\hat{y}_p)$ is robust against P_u and P_h from their guessed values. It also implies that the number of items (N) need not be adjusted, since the sensitivity ratios are very close to one.

The following describes how to determine the sample size that achieves the desired precision of the estimate of a specified quantile at the design stress.

The precision of such an estimate may be specified by requiring that, with a high probability ϕ, the estimate fall within a factor h of the true value θ_0 or t_p. That is,

$$P_r\{\theta_0/h \le \hat{\theta}_0 \le h\theta_0\} \le \phi \ , \qquad (3.6.1)$$

where h(> 1) and ϕ are given constants.
Equation (3.6.1) can be rewritten as

$$P_r\{\mu_0 - \ln h \le \hat{\mu}_0 \le \mu_0 + \ln h\} \ge \phi \ . \qquad (3.6.2)$$

The sample size (N) that approximately achieves this is:

$$N^* \cong v_0[(w/2)/\ln h]^2 \ , \qquad (3.6.3)$$

where v_0 be the asymptotic variance of $\hat{y}_p$ when $N = 1$ and w is the $(1 + \phi)/2$ quantile of the standard normal distribution.

3.7 CONCLUSION

We have developed statistical optimal ALT plans in which two over stress levels are involved under the assumptions of periodic inspection and type-I censoring. The Rayleigh distribution is considered to describe the failure of test units.

We conclude that, the large number of inspections (K) are not needed and estimated failure probabilities are fairly tolerable from their true values of the parameter considered.

For computational purposes, equally spaced inspection times are considered at each stress level. We hope that future research work can be done in this area with other life time distributions and schemes of inspection [see Meecker (1986)].

TABLE 1. OPTIMAL ALT PLANS WHEN P_u = 0.0001

P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{y}_p$)	RATIO
.9	4.605	-5.0221	2	0.708	0.787	41.0588	1.0283
			5	0.710	0.794	40.1125	1.0047
			10	0.710	0.795	39.9712	1.0011
			∞	0.710	0.796	39.9269	1
.5	4.605	-4.4218	2	0.698	0.823	86.4579	1.0029
			5	0.698	0.824	86.2504	1.0005
			10	0.698	0.824	86.2179	1.0001
			∞	0.698	0.824	86.2076	1
.1	4.605	-3.4799	2	0.630	0.849	317.2028	1.0001
			5	0.630	0.849	317.1865	1.0000
			10	0.630	0.849	317.1838	1.0000
			∞	0.630	0.849	317.1830	1
.01	4.605	-2.3050	2	0.444	0.890	1318.6085	1.0000
			5	0.444	0.890	1318.6080	1.0000
			10	0.444	0.890	1318.6079	1.0000
			∞	0.444	0.890	1318.6079	1
.001	4.605	-1.1514	2	0.002	0.999	2501.2496	1.0000
			5	0.002	0.999	2501.2495	1.0000
			10	0.002	0.999	2501.2495	1.0000
			∞	0.002	0.999	2501.2495	1

TABLE 2. OPTIMAL ALT PLANS WHEN P_u = 0.001

P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{y}_p$)	RATIO
.9	3.4536	-3.8707	2	0.620	0.809	23.1310	1.0256
			5	0.624	0.814	22.6495	1.0042
			10	0.624	0.815	22.5773	1.0010
			∞	0.624	0.816	22.5547	1
.5	3.4536	-3.2704	2	0.590	0.847	44.7745	1.0025
			5	0.592	0.846	44.6812	1.0004
			10	0.592	0.846	44.6664	1.0001
			∞	0.592	0.846	44.6618	1
.1	3.4536	-2.3285	2	0.446	0.888	129.8203	1.0000
			5	0.446	0.888	129.8154	1.0000
			10	0.446	0.888	129.8145	1.0000
			∞	0.446	0.888	129.8143	1
.01	3.4536	-1.1536	2	0.002	0.999	250.1682	1.0000
			5	0.002	0.999	250.1682	1.0000
			10	0.002	0.999	250.1682	1.0000
			∞	0.002	0.999	250.1682	1

TABLE 3. OPTIMAL ALT PLANS WHEN P_u = 0.01

P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{y}_p$)	RATIO
.9	2.3001	-2.7171	2	0.458	0.852	10.2933	1.0200
			5	0.464	0.855	10.1247	1.0033
			10	0.466	0.855	10.0993	1.0008
			∞	0.466	0.855	10.0913	1
.5	2.3001	-2.1168	2	0.368	0.898	16.6439	1.0017
			5	0.368	0.899	16.6205	1.0003
			10	0.368	0.899	16.6169	1.0001
			∞	0.368	0.899	16.6157	1
.1	2.3001	-1.1749	2	0.002	0.999	25.0150	1.0000
			5	0.002	0.999	25.0149	1.0000
			10	0.002	0.999	25.0149	1.0000
			∞	0.002	0.999	25.0149	1

TABLE 4. OPTIMAL ALT PLANS WHEN P_u = 0.1

P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{y}_p$)	RATIO
.9	1.1252	-1.5422	2	0.046	0.983	2.4886	1.0031
			5	0.056	0.980	2.4824	1.0006
			10	0.058	0.979	2.4813	1.0001
			∞	0.058	0.979	2.4810	1
.7	1.1252	-1.2180	2	0.002	0.999	2.5033	1.0004
			5	0.002	0.999	2.5024	1.0001
			10	0.002	0.999	2.5023	1.0000
			∞	0.002	0.999	2.5023	1

Table 5. Sensitivities Of avar($\hat{Y}_p$) When P_u = 0.0001

(i) P_h = 0.9

$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.00001	1.1097	1.1007	1.0823	1.0638	1.0446
0.00003	1.0393	1.0341	1.0243	1.0181	1.0114
0.00005	1.0169	1.0149	1.0087	1.0058	1.0067
0.00010	1.0031	1.0014	1	1.0015	1.0126
0.00020	1.0123	1.0087	1.0103	1.0160	1.0360
0.00030	1.0324	1.0273	1.0279	1.0360	1.0618
0.00050	1.0773	1.0658	1.0669	1.0783	1.1072

(ii) P_h = 0.1

$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
0.00001	1.0826	1.1084	1.1267	1.1459	1.1601
0.00003	1.0149	1.0278	1.0391	1.0521	1.0599
0.00005	1.0013	1.0072	1.0146	1.0226	1.0299
0.00010	1.0099	1.0016	1	1.0012	1.0031
0.00020	1.0644	1.0340	1.0178	1.0101	1.0046
0.00030	1.1297	1.0814	1.0524	1.0342	1.0234
0.00050	1.2670	1.1790	1.1301	1.0967	1.0718

(iii) P_h = 0.01

$\tilde{P}_u \backslash \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
0.00003	1.0313	1.0579	1.0843	1.1093	1.1301
0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
0.00010	1.0232	1.0042	1	1.0024	1.0076
0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
0.00030	1.3941	1.2217	1.1359	1.0869	1.0560
0.00050	1.8870	1.5967	1.3890	1.2748	1.1993

(iv) P_h = 0.001

$\tilde{P}_u \backslash \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
0.00010	1.0000	1.0000	1	1.0000	1.0119
0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
0.00050	1.0008	1.0006	1.0004	1.0003	1.0003

Table 06. Sensitivities Of avar($\hat{Y}_p$) When P_u = 0.001

(i) P_h = 0.9

$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.00010	1.1514	1.1484	1.1342	1.1129	1.0873
0.00030	1.0455	1.0464	1.0422	1.0349	1.0259
0.00050	1.0163	1.0178	1.0152	1.0115	1.0112
0.00100	1.0029	1.0011	1	1.0009	1.0085
0.00200	1.0316	1.0232	1.0195	1.0214	1.0348
0.00300	1.0792	1.0596	1.0510	1.0532	1.0700
0.00500	1.1836	1.1477	1.1287	1.1264	1.1390

(ii) P_h = 0.5

$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
0.00010	1.1067	1.1324	1.1540	1.1641	1.1730
0.00030	1.0196	1.0345	1.0476	1.0591	1.0644
0.00050	1.0021	1.0090	1.0174	1.0244	1.0300
0.00100	1.0121	1.0019	1	1.0012	1.0036
0.00200	1.0839	1.0435	1.0229	1.0126	1.0070
0.00300	1.1711	1.1032	1.0641	1.0414	1.0298
0.00500	1.3720	1.2365	1.1648	1.1189	1.0882

(iii) P_h = 0.10000

$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
0.00010	1.1677	1.2187	1.2588	1.2957	1.3208
0.00030	1.0304	1.0567	1.0829	1.1075	1.1279
0.00050	1.0025	1.0151	1.0302	1.0465	1.0639
0.00100	1.0234	1.0043	1	1.0024	1.0077
0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
0.00300	1.3917	1.2221	1.1355	1.0861	1.0552
0.00500	1.9168	1.5895	1.3866	1.2676	1.1925

(iv) P_h = 0.01

$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
0.00010	1.5741	1.7417	1.8847	2.0144	2.1219
0.00030	1.0931	1.1904	1.2813	1.3641	1.4409
0.00050	1.0000	1.0370	1.0951	1.1523	1.2099
0.00100	1.0000	1.0000	1	1.0000	1.0118
0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

Table 7. Sensitivities Of avar($\hat{Y}_p$) When P_u = 0.01

(i) P_h = 0.9

$\tilde{P}_u$ \ $\tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.00100	1.2457	1.2559	1.2527	1.2394	1.2095
0.00300	1.0637	1.0767	1.0821	1.0798	1.0714
0.00500	1.0171	1.0245	1.0292	1.0302	1.0313
0.01000	1.0086	1.0020	1	1.0008	1.0057
0.02000	1.1096	1.0678	1.0411	1.0332	1.0320
0.03000	1.2691	1.1829	1.1229	1.0988	1.0865
0.05000	1.7033	1.4878	1.3327	1.2735	1.2196

(ii) P_h = 0.5

$\tilde{P}_u$ \ $\tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
0.00100	1.2190	1.2838	1.3305	1.3721	1.3991
0.00300	1.0379	1.0733	1.1084	1.1397	1.1624
0.00500	1.0021	1.0184	1.0393	1.0616	1.0837
0.01000	1.0403	1.0069	1	1.0040	1.0129
0.02000	1.2905	1.1338	1.0622	1.0265	1.0094
0.03000	1.4966	1.3468	1.1921	1.1070	1.0590
0.05000	1.4968	1.4967	1.4966	1.3568	1.2251

(iii) P_h = 0.1

$\tilde{P}_u$ \ $\tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
0.00100	1.5684	1.7353	1.8773	2.0058	2.1119
0.00300	1.0945	1.1894	1.2803	1.3631	1.4397
0.00500	1.0000	1.0396	1.0964	1.1564	1.2150
0.01000	1.0000	1.0000	1	1.0000	1.0171
0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
0.05000	1.0008	1.0006	1.0004	1.0003	1.0003

Table 8. Sensitivities of avar($\hat{y}_p$) when $P_u = 0.1$.

(i) $P_h = 0.9$

$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01000	1.6281	1.7080	1.7640	1.7829	1.7784
0.03000	1.1334	1.1918	1.2522	1.2857	1.3134
0.05000	1.0179	1.0502	1.0960	1.1231	1.1594
0.10000	1.0046	1.0046	1	1.0050	1.0231
0.15000	1.0046	1.0046	1.0046	1.0046	1.0012
0.20000	1.0047	1.0047	1.0047	1.0047	1.0047
0.30000	1.0049	1.0048	1.0048	1.0048	1.0048

(ii) $P_h = 0.7$

$\tilde{P}_u \backslash \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
0.01000	1.7232	1.8648	1.9895	2.0972	2.1754
0.03000	1.1220	1.2140	1.3055	1.3960	1.4868
0.05000	1.0000	1.0348	1.0966	1.1670	1.2498
0.10000	1.0000	1.0000	1	1.0000	1.0249
0.15000	1.0001	1.0001	1.0000	1.0000	1.0000
0.20000	1.0002	1.0001	1.0001	1.0001	1.0001
0.30000	1.0004	1.0003	1.0002	1.0002	1.0002

APPENDIX A

STANDARDIZATION OF PARAMETERS

Let us consider use stress $s_0 = 0$ and high stress level $s_2 = 1$. Under this reparameterization, let with and without the prime represent the original and standardized scale, respectively. It has the following transformation:

$$s = (s' - s_0)/(s_2' - s_0') . \tag{3.8.1}$$

It can be written as

$$s' = s(s_2' - s_0') + s_0' . \tag{3.8.2}$$

Next, consider the standardization of time. Let $t_{C1} = t_{C2} = t_C'$ in the original time scale. In the standardized scale, every parameter whose unit is time must be divided by t_C' i.e. $\theta = \theta'/t_C'$. Therefore, we have the following relationship:

$$\begin{aligned}\theta &= e^{(\beta_0' + \beta_1' s')} / t_C' \\ &= e^{(\beta_0' + \beta_1'\{s(s_2' - s_0') + s_0'\})} e^{-\ln t_C'} \\ &= e^{\{(\beta_0' + \beta_1' s_0' - \ln t_C') + \beta_1' s(s_2' - s_0')\}} .\end{aligned}$$

Since $\theta = e^{(\beta_0 + \beta_1 s)}$.
We have

$$\beta_0 = \beta_0' + \beta_1' s_0' - \ln t_C' , \tag{3.8.3}$$

$$\beta_1 = \beta_1'(s_2' - s_0') , \tag{3.8.4}$$

or, equivalently,

$$\beta_1' = \beta_1/(s_2' - s_0') , \tag{3.8.5}$$

and

$$\beta_0' = \beta_0 + \ln t_c' - \beta_1' s_0'$$
$$= \beta_0 + \ln t_c' - \beta_1 s_0'/(s_2' - s_0') \quad . \tag{3.8.6}$$

From equation (3.2.9) and (3.8.3)

$$\hat{y}_p' = \frac{1}{2}\ln 2 + \hat{\beta}_0' + \hat{\beta}_1' s_0' + \frac{1}{2}\ln\{-\ln(1-p)\}$$
$$= \frac{1}{2}\ln 2 + \hat{\beta}_0 + \ln t_c' + \frac{1}{2}\ln\{-\ln(1-p)\} \quad . \tag{3.8.7}$$

Also,

$$\hat{y}_p = \frac{1}{2}\ln 2 + \hat{\beta}_0 + \frac{1}{2}\ln\{-\ln(1-p)\} \quad . \tag{3.8.8}$$

It can be shown that

$$\text{avar}(\hat{y}_p') = \text{avar}\ (\hat{y}_p)$$

Therefore, no generality is lost under the above standardization.

APPENDIX B

DESCRIPTION OF TWO-STEP PROCEDURE

The two-step procedure was adopted to optimize avar($\hat{y}_p$) respect to s_1 and α_1. This technique is applied to our problem according to the following steps:
(1) We optimize α_1 (say α_1^*).
That is, from equation (3.5.4)

$$\frac{\partial(\text{avar}(\hat{y}_p))}{\partial\alpha_1} = N^{-1}\{s_1^2Q_1 - Q_2)\alpha_1^2 + 2Q_2\alpha_1 - Q_2\} \Big/ \{Q_1Q_2(s_1 - 1)^2(-\alpha_1^2 + \alpha_1)^2\} = 0 \quad (3.9.1)$$

The optimal value of α_1, for $0 < \alpha_1 < 1$, is given by

$$\alpha_1^* = \left(-Q_2 + \sqrt{s_1^2\, Q_1Q_2}\,\right) \Big/ \left(s_1^2\, Q_1 - Q_2\right) \quad (3.9.2)$$

(2) We employ the grid search with respect to s_1 to optimize avar($\hat{y}_p$). That is, α_1^* and the minimum of avar($\hat{y}_p$) are determined on the grid s_1 = d, 2d, 3d, ..., where d is the grid size. Finally, minimum value of avar($\hat{y}_p$) is determined by optimal s_1 (i.e., s_1^*) and the corresponding α_1^* among all the grid points considered.

In actual computational experiments, the above method was tried for a given set of P_u, P_h (and corresponding β_0, β_1) and K with 500 different grid points. Besides, the grid size for s_1 is set to 0.002. Finally, N is set to 1, since it only as a scale factor for avar($\hat{y}_p$) the optimal solution consists of s_1^* (and the corresponding α_1^*) for which avar($\hat{y}_p$) attains the minimum among all the grid points considered.

CHAPTER 4

OPTIMAL DESIGN OF ACCELERATED LIFE TESTS FOR THE WEIBULL DISTRIBUTION UNDER PERIODIC INSPECTION AND TYPE I CENSORING.

4.1 INTRODUCTION

Accelerated life testing (ALT) reduces testing time resulting in the reduction of the cost of conducting the test. Further reduction in time and cost is possible through periodic inspection. Earlier studies assumed continuous inspection for ALT, studies on statistical analysis of the data generated by periodic inspection were largely concerned with life testing at 'normal' or 'use' conditions. Yum and Choi (1989) first attempted to combine these interesting and important features of life tests, namely, acceleration and periodic inspection. They developed an asymptotically optimal ALT plan for the exponentially distributed lifetimes under the assumption of Type I censoring and periodic inspection. Later Seo and Yum (1991) extended Yum and Choi (1989) work by considering Weibull distribution to describe the lifetimes of test units. However, they have used $Y = \ln T$, which has an extreme value distribution.

This study is a generalization of Yum and Choi work. Weibull distribution have been considered to describe the lifetime distribution of the units under test. Two stress levels have been considered and the tests are type I censored and the periodic inspection is at equally spaced (ES) time period. A self developed software has been used to carry out the computations of asymptotic variance of ML estimator. Sensitivity analysis have also been carried out to assess the effect of inaccuracy in the 'guesstimates' of the unknown parameters on the optimal plan.

For the shape parameter $\delta = 1$, the Weibull distribution reduces to exponential. In our computation, asymptotic variances obtained by Yum and Choi (1989), follow for $\delta = 1$. Although there is a slight difference in the corresponding values for sensitivity analysis, the essential feature (i.e. the trend) is the same.

Computational experiments for various combination of parameters also show how optimal plans vary with respect to the parameters involved.

4.2 THE MODEL

Let us consider three stress levels s_0, s_1 and s_2. The use stress level s_0 and the high stress level s_2 are assumed to be known and the low stress level s_1 $(s_0 < s_1 < s_2)$ is to be optimally determined.

At these stress levels, we assume that the lifetimes(T) of test units identically and independently follow a Weibull distribution, given as

$$f(t/\theta,\delta) = (\delta/\theta)\left(t/\theta\right)^{\delta-1}e^{-(t/\theta)^{\delta}}, \quad t > 0 . \qquad (4.2.1)$$

Also, assume that the scale parameter θ and the stress s are related as

$$\theta = e^{(\beta_0 + \beta_1 s)} , \qquad (4.2.2)$$

where β_0, β_1 and δ are parameters of the product and test method.

If s is the log of voltage stress, then (4.2.2) is the inverse power law (i.e.,Meeker and Nelson (1975); Nelson (1990);Nelson and Meeker (1978)). If s is reciprocal of absolute temperature, then (4.2.2) is the Arrhenius relationship (i.e,Kielpinski and Nelson (1975); Nelson (1990); Nelson and Kielpinski(1976)). Other relationships are presented by (i.e., Chernoff(1962); Mann, Schafer and Singapurwalla (1974); Nelson (1990)).

The number of test items allocated to s_1 and s_2 are, respectively, given by

$$n_1 = \alpha_1 N, \ n_2 = \alpha_2 N , \tag{4.2.3}$$

where $\alpha_1 + \alpha_2 = 1$, N is the total number of test items and α_1 is to be optimally determined.

Let at stress level s_i, n_i units are to be put on test at time 0 and run until a pre-specified time t_{ci} and inspections are conducted only at specified points in time t_{i1}, t_{i2}, ..., $t_{i,K(i)}$, where $t_{i,K(i)} = t_{ci}$.

Also, let $t_{io} = 0$ and $t_{i,K(i)+1} = \infty$, and define at stress level s_i

x_{ij} = the number of items failed in $(t_{i,j-1}, t_{ij})$,

p_{ij} = the probability of failure in $(t_{i,j-1}, t_{ij})$,

where $j = 1,2,\ldots,K(i)+1$.

Prob. of Failure	p_{i1}	p_{i2}	...	$p_{i,K(i)}$	$p_{i,K(i)+1}$
No. of Failure	x_{i1}	x_{i2}	...	$x_{i,K(i)}$	$x_{i,K(i)+1}$

$t_{io} = 0$ $\quad t_{i1}$ $\quad t_{i2}$ $\quad t_{i,K(i)-1}$ $\quad t_{i,K(i)} = t_{ci}$ $\quad t_{i,K(i)+1} = \infty$

Figure 1. Structure of periodic inspection at the i^{th} stress level.

The grouped data $\{x_{ij}, \ i = 1, 2; \ j = 1, 2, \ldots, K(i)+1\}$ are used to estimate β_0 and β_1. The estimated relationship is then extrapolated to estimate some quantities at the use condition.

Our particular interest is in the logarithm of the mean life time at the use condition, is given by

$$\mu_0 = \ln \theta_0 = \beta_0 + \beta_1 s_0 . \tag{4.2.4}$$

(4.2.4) is equivalent to (4.2.2) when $\mu = \ln\theta$.

Now, let t_p be the p^{th} quantile of the Weibull distribution at the use condition, given by,

$$t_p = \theta_0\{-\ln(1-p)\}^{1/\delta} . \tag{4.2.5}$$

$$y_p = \ln t_p = \beta_0 + \beta_1 s_0 + (1/\delta).\ln\{-\ln(1-p)\} . \tag{4.2.6}$$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ are the ML estimators of β_0 and β_1, respectively. Then,

$$\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 s_0 . \tag{4.2.7}$$

$$\hat{y}_p = \hat{\mu}_0 + (1/\delta).\ln\{-\ln(1-p)\} . \tag{4.2.8}$$

The problem of optimal ALT plans under periodic inspection can now be stated as: given N, s_0, s_2, $\{t_{ij}$, $i = 1,2$; $j = 1,2,\ldots,K(i)\}$ determine α_1 and s_1 such that the variance of $\hat{\mu}_0$ (or $\hat{y}_p$) is minimized.

4.3 MAXIMUM LIKELIHOOD ESTIMATION

At s_i, the grouped data $\{x_{ij}$, $j = 1,2,\ldots,K(i)+1\}$ are multinomially distributed with parameters n_i and $\{P_{ij}$, $j = 1,2,\ldots,K(i)+1\}$. The likelihood function is given by

$$\begin{aligned} L' &= \prod_{i=1}^{2} L'_i \\ &= \prod_{i=1}^{2} n_i! \left[\prod_{j=1}^{K(i)+1} x_{ij}!\right]^{-1} . \left[\prod_{j=1}^{K(i)+1} P_{ij}{}^{x_{ij}}\right] . \end{aligned} \tag{4.3.1}$$

Taking logarithm of L' yields

$$L = \ln L' = \sum_{i=1}^{2} \ln L_i'$$

$$= C + \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} x_{ij} \ln P_{ij} , \qquad (4.3.2)$$

where C is a constant with respect to β_0 and β_1 and for i = 1, 2 and j = 1,2,...,K(i)+1,

$$P_{ij} = \int_{t_{i,j-1}}^{t_{ij}} f(t)dt = F(t_{ij}) - F(t_{i,j-1})$$

$$= e^{-(t_{i,j-1}/\theta_i)^\delta} - e^{-(t_{i,j}/\theta_i)^\delta} . \qquad (4.3.3)$$

Therefore,

$$L=C+\sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} x_{ij} \ln\left(e^{-(t_{i,j-1}/\theta_i)^\delta} - e^{-(t_{i,j}/\theta_i)^\delta}\right), \qquad (4.3.4)$$

Then, the M L estimates of β_0 and β_1 are obtained by solving the following equations.

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} \left(x_{ij}(B_{i,j-1} - B_{ij})/P_{ij}\right) = 0 , \qquad (4.3.5)$$

$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{2} s_i \sum_{j=1}^{K(i)+1} \left(x_{ij}(B_{i,j-1} - B_{ij})/P_{ij}\right) = 0 , \qquad (4.3.6)$$

where

$$B_{ij} = \delta.(t_{ij}/\theta_i)^{\delta}\ e^{-(t_{ij}/\theta_i)^{\delta}}\ , \tag{4.3.7}$$
for $i=1,2,$ and $j=1,2,\ldots,K(i)+1$.

These estimates can be found out by Monte Carlo studies.

4.4 ASYMPTOTIC CO-VARIANCE OF THE M L ESTIMATOR

The Fisher information matrix F, is defined as

$$F = N.(f_{gh})\ ;\ g,\ h = 0,\ 1\ , \tag{4.4.1}$$

$$f_{gh} = \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{K(i)+1} \left(\frac{\partial P_{ij}}{\partial \beta_g}\right)\left(\frac{\partial P_{ij}}{\partial \beta_h}\right) / P_{ij}\ . \tag{4.4.2}$$

After some algebraic manipulation, we get

$$f_{00} = \sum_{i=1}^{2} \alpha_i Q_i, \tag{4.4.3}$$

$$f_{01} = f_{10} = \sum_{i=1}^{2} \alpha_i s_i Q_i, \tag{4.4.4}$$

$$f_{11} = \sum_{i=1}^{2} \alpha_i s_i^2 Q_i\ , \tag{4.4.5}$$

where

$$Q_i = \sum_{j=1}^{K(i)+1} (B_{i,j-1} - B_{ij})^2 / P_{ij}\ . \tag{4.4.6}$$

The asymptotic covariance matrix of $\hat{\beta}_0$ and $\hat{\beta}_1$ is given by

F^{-1}. That is,

$$V = F^{-1} = N^{-1} \cdot \begin{bmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{bmatrix}$$

$$= N^{-1}\left(f_{00}f_{11} - f_{01}^2\right) - \begin{bmatrix} f_{11} & -f_{01} \\ -f_{10} & f_{00} \end{bmatrix} . \qquad (4.4.7)$$

Then, the asymptotic variance (avar) of $\hat{Y}_p$ or $\hat{\mu}_0$ is given by

$$\begin{aligned} \text{avar}(\hat{\mu}_0) &= \text{avar}(\hat{\beta}_0 + \hat{\beta}_1 s_0) \\ &= N^{-1}(v_{00} + s_0^2 v_{11} + 2s_0 v_{01}) \\ &= N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}(f_{11} + s_0^2 f_{00} - 2s_0 f_{01}) . \qquad (4.4.8) \end{aligned}$$

Which is also avar($\hat{y}_p$).

4.5 STATISTICALLY OPTIMAL PLANS

The optimization problem is to determine s_1 and α_1 such that avar($\hat{\mu}_0$) is minimized under the following assumptions.

(1) Two over stress levels are considered, one is the low stress level(s_1) and the other is the high stress level(s_2) where s_2 is pre-specified.

(2) Censoring times at s_1 and s_2 are same, that is, $t_{C1} = t_{C2} = t_C$.

(3) The number of inspections at each stress level is same, that is, $K(1) = K(2) = K$.

(4) The use stress level $s_0 = 0$

Then equation (4.4.8) is reduced to

$$\text{avar}(\hat{\mu}_0) = N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1} f_{11} \qquad (4.5.1)$$

After some algebraic manipulation we obtain

$$\text{avar}(\hat{\mu}_0) = N^{-1}(s_2^2Q_2 + (s_1^2Q_1 - s_2^2Q_2)\alpha_1)/ \\ (Q_1Q_2(s_1 - s_2)^2(-\alpha_1^2 + \alpha_1)) \quad (4.5.2)$$

We have adopted two-step procedure to minimize the avar($\hat{\mu}_0$) with respect to s_1 and α_1.

Firstly, we optimize α_1 for given s_2. The optimal value α_1^* of α_1, is given by

$$\alpha_1^* = \left(-s_2^2Q_2 + \sqrt{s_1^2s_2^2Q_1Q_2}\right)/\left(s_1^2Q_1 - s_2^2Q_2\right) \quad (4.5.3)$$

Secondly, we optimize with respect to s_1 by grid search. That is, α_1^* and the minimum of avar($\hat{\mu}_0$) are obtained on the grid s_1 = d, 2d, 3d, ..., where d is the grid size. Finally, the optimal plan consists of s_1^* (and the corresponding α_1^*) for which avar($\hat{\mu}_0$) attains the minimum among all the grid points considered.

When K(1) and K(2) go to infinity (i.e. continuous inspection), then optimal s_1^* , α_1^* and corresponding minimum avar($\hat{\mu}_0$) are determined using the method described by Nelson and Meeker (1978) and can also be determined by the above two-step procedure.

4.6 COMPUTATIONAL RESULTS AND DISCUSSIONS

For the purpose of computational experiments, we standardize the parameters such that the common censoring time is 1, use stress $s_0 = 0$ and high stress $s_2 = 1$.

Under the above reparameterization the original and the adjusted parameters will be denoted with and without the prime, respectively, having the following relationship:

$$s' = s(s_2' - s_0') + s_0' . \quad (4.6.1)$$

$$\beta_1' = \beta_1/(s_2' - s_0') \ . \tag{4.6.2}$$

$$\beta_0' = \beta_0 + \ln t_C' - \beta_1 s_0'/(s_2' - s_0') \ . \tag{4.6.3}$$

Let $\mu_0' = \beta_0' + \beta_1' s_0'$. Then it can be shown that

$$\begin{aligned} \text{avar}(\hat{\mu}_0') &= \text{avar}(\hat{\beta}_0' + \hat{\beta}_1' s_0') \\ &= \text{avar}(\hat{\beta}_0) \\ &= \text{avar}\ (\hat{\mu}_0) \ . \end{aligned} \tag{4.6.4}$$

This shows that standardization does not alter the nature of the problem.

In actual experiments, we use the 'guesstimates' of the following quantities instead of β_0 and β_1.

$P_u = P_r\{$an item fails at the use condition in $(0,\ t_C = 1)\}$

$P_h = P_r\{$an item fails at the higher stress in $(0,\ t_C = 1)\}$

Then, the corresponding β_0 and β_1 determined as follows:

$$\beta_0 = \frac{1}{\delta} \ln\{-1/\ln(1 - P_u)\} \tag{4.6.5}$$

$$\beta_1 = \frac{1}{\delta} \ln\{\ln(1 - P_u)/\ln(1 - P_h)\} \tag{4.6.6}$$

Besides, the number of grid points for s_1 is set to 500, therefore the grid size will be 0.002, since $0 < s_1 < 1$ and N is set to 1 since it only serves as a scale factor for $\text{avar}(\hat{\mu}_0)$.

For given P_u and P_h,we calculate the corresponding β_0 and β_1 using eqs (4.6.5) and (4.6.6), respectively.Then minimum $\text{avar}(\hat{\mu}_0)$ is determined from equation (4.5.2) and optimal values of s_1 and α_1 are obtained by applying two-step procedure for equally spaced (ES) inspection times for various values of K (see Appendix A).

For some selected guessed values of P_u and P_h, a sensitivity analysis is conducted.

Computational results are summarized in Tables 1-5, elaborating statistically optimal plans and values of avar($\hat{\mu}_0$) for various combinations of δ, P_u, P_h and K. Ratios are determined by the formula avar($\hat{\mu}_0$(K))/avar($\hat{\mu}_0(\infty)$). We observe that β_0 decreases and β_1 increases as shape parameter δ increases. We also note that when P_u increases, β_0 decreases and β_1 increases. For fixed P_u, decrease in P_h implies increase in β_1. Secondly, avar($\hat{\mu}_0$) for K = 2, 5 or 10, is sufficiently close to avar($\hat{\mu}_0$) for K = ∞, and therefore, unnecessarily large K is not needed. The detailed observations are given in section 4.7. We also note that s_1^* and α_1^* are fairly stable over K.

Finally, as P_u increases and/or P_h decreases s_1^* gets close to s_0 and α_1^* to 1 for each value of shape parameter. Also when $P_u = 0.1$, $s_1^* \cong 0$ and $\alpha_1^* \cong 1$ for $P_h \leq 0.9$, which implies almost no need for an ALT. We observe similar trends when P_h values are small for $P_u < 0.1$.

Tables 6-10 show sensitivity analysis for different value of shape parameter. Sensitivity analysis is conducted by using guessed values of P_u and P_h as $\tilde{P}_u$ and $\tilde{P}_h$, respectively. For these guessed values the optimal s_1^* is determined as $\tilde{s}_1^*$ and optimal α_1^* as $\tilde{\alpha}_1^*$. Finally, avar($\hat{\mu}_0(\tilde{s}_1^*,\tilde{\alpha}_1^*)$) is calculated. Then, sensitivity is calculated as the ratio of avar($\hat{\mu}_0(\tilde{s}_1^*,\tilde{\alpha}_1^*)$) to avar($\hat{\mu}_0(s_1^*,\alpha_1^*)$) for various cases with K = 2. We note that these ratios are very close to 1 implying that the avar($\hat{\mu}_0$) is robust against the true P_u, P_h and δ from their guessed values over K.

Now we select a sample of size N by the method described by Meeker (1986). Let v_0 be the asymptotic variance of $\hat{\mu}_0$ when N = 1. The sample size N is given by

$$N^* \cong \frac{v_0 W^2}{(\ln h)^2\ \delta^2} \qquad (4.6.7)$$

where W is the $(1 + \phi)/2$ quantile of standard normal distribution.

4.7 CONCLUSION

We have presented, how the statistically optimal ALT plans are developed when the failures are observed under periodic inspection and Type-I censoring for the case of Weibull distributed lifetimes in which two overstress levels are involved.

Computational results indicate that:

(1) asymptotic variance of $\hat{\mu}_0$ decreases when shape parameter δ increases.

(2) avar($\hat{\mu}_0$) decreases as P_u increases and it becomes minimum when P_u equals 0.1.

(3) When P_h decreases, asymptotic variance of $\hat{\mu}_0$ increases for fixed P_u.

(4) asymptotic variance of $\hat{\mu}_0$ decreases as P_u increases and/or P_h decreases over K.

(5) For fixed P_u and P_h, there is no significance difference in asymptotic variance of $\hat{\mu}_0$ for variation in K, therefore we need not have large number of inspections.

(6) When K goes to infinity, asymptotic variance of $\hat{\mu}_0$ is almost the same as it is for $K = 10$. This implies that the number of inspections more than 10 are totally unnecessary.

(7) For each value of shape parameter (δ), sensitivity analysis shows that the avar($\hat{\mu}_0$) is robust against the true P_u and P_h from their guessed values for all K.

TABLE 1. OPTIMAL ALT PLANS WHEN δ = 0.5

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N avar($\hat{\mu}_0$)	RATIO
0.0001	.9	18.420	-20.0883	2	0.706	0.784	665.8666	1.0415
				5	0.710	0.790	647.5179	1.0128
				10	0.710	0.794	642.3535	1.0047
				∞	0.710	0.795	639.3316	1
	.5	18.420	-17.6872	2	0.698	0.823	1383.3530	1.0029
				5	0.698	0.824	1380.4724	1.0008
				10	0.698	0.824	1379.7507	1.0003
				∞	0.698	0.824	1379.3707	1
	.1	18.420	-13.9195	2	0.630	0.849	5075.2080	1.0001
				5	0.630	0.849	5075.0034	1.0000
				10	0.630	0.849	5074.9551	1.0000
				∞	0.630	0.849	5074.9309	1
	.01	18.420	-9.2199	2	0.444	0.890	21097.7340	1.0000
				5	0.444	0.890	21097.7282	1.0000
				10	0.444	0.890	21097.7269	1.0000
				∞	0.444	0.890	21097.7262	1
	.001	18.420	-4.6057	2	0.002	0.999	40019.9937	1.0000
				5	0.002	0.999	40019.9934	1.0000
				10	0.002	0.999	40019.9934	1.0000
				∞	0.002	0.999	40019.9934	1
0.001	.9	13.814	-15.482	2	0.618	0.806	374.6206	1.0374
				5	0.622	0.812	365.2967	1.0115
				10	0.624	0.814	362.6713	1.0043
				∞	0.624	0.816	361.1303	1
	.5	13.814	-13.0815	2	0.590	0.847	716.4038	1.0025
				5	0.590	0.847	715.1086	1.0007
				10	0.592	0.846	714.7829	1.0002
				∞	0.592	0.846	714.6106	1
	.1	13.814	-9.3138	2	0.446	0.888	2077.1143	1.0000
				5	0.446	0.888	2077.0517	1.0000
				10	0.446	0.888	2077.0369	1.0000
				∞	0.446	0.888	2077.0295	1
	.01	13.814	-4.6142	2	0.002	0.999	4002.6909	1.0000
				5	0.002	0.999	4002.6908	1.0000
				10	0.002	0.999	4002.6908	1.0000
				∞	0.002	0.999	4002.6908	1
0.01	.9	9.200	-10.8684	2	0.456	0.849	166.2536	1.0291
				5	0.462	0.854	163.0091	1.0090
				10	0.464	0.855	162.0910	1.0033
				∞	0.466	0.855	161.5510	1
	.5	9.200	-8.4673	2	0.368	0.898	266.3049	1.0017
				5	0.368	0.899	265.9805	1.0005
				10	0.368	0.899	265.8993	1.0002
				∞	0.368	0.899	265.8566	1
	.1	9.200	-4.6999	2	0.002	0.999	400.2400	1.0000
				5	0.002	0.999	400.2390	1.0000
				10	0.002	0.999	400.2388	1.0000
				∞	0.002	0.999	400.2387	1
0.1	.9	4.500	-6.1688	2	0.042	0.984	39.8574	1.0040
				5	0.052	0.981	39.7526	1.0013
				10	0.056	0.980	39.7194	1.0005
				∞	0.058	0.979	39.6993	1
	.7	4.500	-4.8720	2	0.002	0.999	40.0519	1.0004
				5	0.002	0.999	40.0403	1.0001
				10	0.002	0.999	40.0376	1.0000
				∞	0.002	0.999	40.0362	1

TABLE 2. OPTIMAL ALT PLANS WHEN δ = 0.9

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{\mu}_o$)	RATIO
0.0001	.9	10.233	-11.160	2	0.708	0.788	202.1383	1.0252
				5	0.710	0.794	197.9871	1.0042
				10	0.710	0.795	197.3668	1.0010
				∞	0.710	0.796	197.1688	1
	.5	10.233	-9.8262	2	0.698	0.823	426.5117	1.0019
				5	0.698	0.824	425.8439	1.0003
				10	0.698	0.824	425.7462	1.0001
				∞	0.698	0.824	425.7153	1
	.1	10.233	-7.7331	2	0.630	0.849	1566.3937	1.0000
				5	0.630	0.849	1566.3454	1.0000
				10	0.630	0.849	1566.3384	1.0000
				∞	0.630	0.849	1566.3362	1
	.01	10.233	-5.1222	2	0.444	0.890	6511.6489	1.0000
				5	0.444	0.890	6511.6475	1.0000
				10	0.444	0.890	6511.6473	1.0000
				∞	0.444	0.890	6511.6472	1
	.001	10.233	-2.5587	2	0.002	0.999	12351.8558	1.0000
				5	0.002	0.999	12351.8556	1.0000
				10	0.002	0.999	12351.8556	1.0000
				∞	0.002	0.999	12351.8556	1
0.001	.9	7.674	-8.6014	2	0.620	0.810	113.9112	1.0227
				5	0.624	0.814	111.7980	1.0037
				10	0.624	0.815	111.4816	1.0009
				∞	0.624	0.816	111.3804	1
	.5	7.674	-7.2675	2	0.590	0.847	220.9104	1.0016
				5	0.592	0.846	220.6097	1.0003
				10	0.592	0.846	220.5654	1.0001
				∞	0.592	0.846	220.5514	1
	.1	7.674	-5.1743	2	0.446	0.888	641.0759	1.0000
				5	0.446	0.888	641.0611	1.0000
				10	0.446	0.888	641.0590	1.0000
				∞	0.446	0.888	641.0583	1
	.01	7.674	-2.5635	2	0.002	0.999	1235.3987	1.0000
				5	0.002	0.999	1235.3987	1.0000
				10	0.002	0.999	1235.3987	1.0000
				∞	0.002	0.999	1235.3987	1
0.01	.9	5.111	-6.0380	2	0.460	0.851	50.7185	1.0178
				5	0.464	0.855	49.9801	1.0029
				10	0.466	0.855	49.8691	1.0007
				∞	0.466	0.855	49.8334	1
	.5	5.111	-4.7040	2	0.368	0.899	82.1424	1.0011
				5	0.368	0.899	82.0671	1.0002
				10	0.368	0.899	82.0561	1.0000
				∞	0.368	0.899	82.0527	1
	.1	5.111	-2.6109	2	0.002	0.999	123.5307	1.0000
				5	0.002	0.999	123.5305	1.0000
				10	0.002	0.999	123.5305	1.0000
				∞	0.002	0.999	123.5305	1
0.1	.9	2.500	-3.4271	2	0.048	0.982	12.2838	1.0026
				5	0.056	0.980	12.2575	1.0005
				10	0.058	0.979	12.2532	1.0001
				∞	0.058	0.979	12.2518	1
	.7	2.500	-2.7067	2	0.002	0.999	12.3600	1.0003
				5	0.002	0.999	12.3573	1.0000
				10	0.002	0.999	12.3569	1.0000
				∞	0.002	0.999	12.3568	1

TABLE 3. OPTIMAL ALT PLANS WHEN δ = 1.0

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N avar($\hat{\mu}_0$)	RATIO
0.0001	.9	9.210	-10.0442	2	0.708	0.789	163.4067	1.0232
				5	0.710	0.794	160.2807	1.0036
				10	0.710	0.795	159.8415	1.0009
				∞	0.710	0.796	159.7047	1
	.5	9.210	-8.8436	2	0.698	0.823	345.4500	1.0018
				5	0.698	0.824	344.9270	1.0003
				10	0.698	0.824	344.8525	1.0001
				∞	0.698	0.824	344.8292	1
	.1	9.210	-6.9598	2	0.630	0.849	1268.7776	1.0000
				5	0.630	0.849	1268.7391	1.0000
				10	0.630	0.849	1268.7336	1.0000
				∞	0.630	0.849	1268.7319	1
	.01	9.210	-4.6100	2	0.444	0.890	5274.4328	1.0000
				5	0.444	0.890	5274.4317	1.0000
				10	0.444	0.890	5274.4315	1.0000
				∞	0.444	0.890	5274.4315	1
	.001	9.210	-2.3029	2	0.002	0.999	10004.9984	1.0000
				5	0.002	0.999	10004.9983	1.0000
				10	0.002	0.999	10004.9983	1.0000
				∞	0.002	0.999	10004.9983	1
0.001	.9	3.453	-3.8707	2	0.620	0.809	23.1310	1.0256
				5	0.624	0.814	22.6495	1.0042
				10	0.624	0.815	22.5773	1.0010
				∞	0.624	0.816	22.5547	1
	.5	3.453	-3.2704	2	0.590	0.847	44.7745	1.0025
				5	0.592	0.846	44.6812	1.0004
				10	0.592	0.846	44.6664	1.0001
				∞	0.592	0.846	44.6618	1
	.1	3.453	-2.3285	2	0.446	0.888	129.8203	1.0000
				5	0.446	0.888	129.8154	1.0000
				10	0.446	0.888	129.8145	1.0000
				∞	0.446	0.888	129.8143	1
	.01	3.453	-1.1536	2	0.002	0.999	250.1682	1.0000
				5	0.002	0.999	250.1682	1.0000
				10	0.002	0.999	250.1682	1.0000
				∞	0.002	0.999	250.1682	1
0.01	.9	4.600	-5.4342	2	0.460	0.852	41.0247	1.0164
				5	0.464	0.855	40.4681	1.0026
				10	0.466	0.855	40.3894	1.0006
				∞	0.466	0.855	40.3647	1
	.5	4.600	-4.2336	2	0.368	0.899	66.5326	1.0011
				5	0.368	0.899	66.4737	1.0002
				10	0.368	0.899	66.4653	1.0000
				∞	0.368	0.899	66.4626	1
	.1	4.600	-2.3498	2	0.002	0.999	100.0599	1.0000
				5	0.002	0.999	100.0597	1.0000
				10	0.002	0.999	100.0597	1.0000
				∞	0.002	0.999	100.0597	1
0.1	.9	2.250	-3.0844	2	0.048	0.982	9.9481	1.0024
				5	0.056	0.980	9.9280	1.0004
				10	0.058	0.979	9.9249	1.0001
				∞	0.058	0.979	9.9240	1
	.7	2.250	-2.4360	2	0.002	0.999	10.0116	1.0003
				5	0.002	0.999	10.0094	1.0000
				10	0.002	0.999	10.0091	1.0000
				∞	0.002	0.999	10.0090	1

TABLE 4. OPTIMAL ALT PLANS WHEN δ = 3.0

P_u	P_h	β_o	β_1	K	S_1^*	a_1^*	N avar($\hat{\mu}_o$)	RATIO
0.0001	.9	3.0700	-3.3481	2	0.704	0.783	18.6304	1.0498
				5	0.710	0.793	17.8906	1.0081
				10	0.710	0.795	17.7809	1.0019
				∞	0.710	0.796	17.7463	1
	.5	3.0700	-2.9479	2	0.696	0.824	38.4914	1.0046
				5	0.698	0.824	38.3497	1.0009
				10	0.698	0.824	38.3233	1.0002
				∞	0.698	0.824	38.3147	1
	.1	3.0700	-2.3199	2	0.630	0.849	140.9838	1.0001
				5	0.630	0.849	140.9731	1.0000
				10	0.630	0.849	140.9710	1.0000
				∞	0.630	0.849	140.9703	1
	.01	3.0700	-1.5367	2	0.444	0.890	586.0484	1.0000
				5	0.444	0.890	586.0481	1.0000
				10	0.444	0.890	586.0481	1.0000
				∞	0.444	0.890	586.0480	1
	.001	3.0700	-0.7676	2	0.002	0.999	1111.6668	1.0000
				5	0.002	0.999	1111.6667	1.0000
				10	0.002	0.999	1111.6667	1.0000
				∞	0.002	0.999	1111.6667	1
0.001	.9	2.3024	-2.5804	2	0.616	0.805	10.4744	1.0448
				5	0.624	0.813	10.0985	1.0074
				10	0.624	0.815	10.0425	1.0018
				∞	0.624	0.816	10.0248	1
	.5	2.3024	-2.1803	2	0.590	0.846	19.9295	1.0040
				5	0.590	0.847	19.8656	1.0008
				10	0.592	0.846	19.8537	1.0002
				∞	0.592	0.846	19.8498	1
	.1	2.3024	-1.5523	2	0.446	0.888	57.6994	1.0001
				5	0.446	0.888	57.6961	1.0000
				10	0.446	0.888	57.6955	1.0000
				∞	0.446	0.888	57.6953	1
	.01	2.3024	-0.7690	2	0.002	0.999	111.1859	1.0000
				5	0.002	0.999	111.1859	1.0000
				10	0.002	0.999	111.1859	1.0000
				∞	0.002	0.999	111.1859	1
0.01	.9	1.5334	-1.8114	2	0.454	0.848	4.6420	1.0350
				5	0.464	0.854	4.5111	1.0058
				10	0.464	0.856	4.4915	1.0014
				∞	0.466	0.855	4.4852	1
	.5	1.5334	-1.4112	2	0.366	0.899	7.4047	1.0027
				5	0.368	0.899	7.3887	1.0005
				10	0.368	0.899	7.3858	1.0001
				∞	0.368	0.899	7.3848	1
	.1	1.5334	-0.7833	2	0.002	0.999	11.1178	1.0000
				5	0.002	0.999	11.1178	1.0000
				10	0.002	0.999	11.1177	1.0000
				∞	0.002	0.999	11.1177	1
0.1	.9	0.7501	-1.0281	2	0.038	0.985	1.1081	1.0049
				5	0.054	0.981	1.1037	1.0010
				10	0.058	0.979	1.1029	1.0002
				∞	0.058	0.979	1.1027	1
	.7	0.7501	-0.8120	2	0.002	0.999	1.1129	1.0007
				5	0.002	0.999	1.1123	1.0001
				10	0.002	0.999	1.1122	1.0000
				∞	0.002	0.999	1.1121	1

TABLE 5. OPTIMAL ALT PLANS WHEN δ = 4.0

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N avar($\hat{\mu}_o$)	RATIO
0.0001	.9	2.3025	-2.5110	2	0.702	0.778	10.6707	1.0689
				5	0.710	0.791	10.1110	1.0128
				10	0.710	0.795	10.0140	1.0031
				∞	0.710	0.796	9.9830	1
	.5	2.3025	-2.2109	2	0.696	0.823	21.6769	1.0058
				5	0.698	0.823	21.5833	1.0014
				10	0.698	0.824	21.5601	1.0004
				∞	0.698	0.824	21.5523	1
	.1	2.3025	-1.7399	2	0.630	0.849	79.3051	1.0001
				5	0.630	0.849	79.2983	1.0000
				10	0.630	0.849	79.2964	1.0000
				∞	0.630	0.849	79.2958	1
	.01	2.3025	-1.1525	2	0.444	0.890	329.6522	1.0000
				5	0.444	0.890	329.6520	1.0000
				10	0.444	0.890	329.6520	1.0000
				∞	0.444	0.890	329.6520	1
	.001	2.3025	-0.5757	2	0.002	0.999	625.3124	1.0000
				5	0.002	0.999	625.3124	1.0000
				10	0.002	0.999	625.3124	1.0000
				∞	0.002	0.999	625.3124	1
0.001	.9	1.7268	-1.9353	2	0.614	0.800	5.9885	1.0619
				5	0.622	0.813	5.7046	1.0116
				10	0.624	0.815	5.6551	1.0028
				∞	0.624	0.816	5.6393	1
	.5	1.7268	-1.6352	2	0.590	0.846	11.2219	1.0050
				5	0.590	0.847	11.1796	1.0013
				10	0.592	0.846	11.1692	1.0003
				∞	0.592	0.846	11.1656	1
	.1	1.7268	-1.1642	2	0.446	0.888	32.4564	1.0001
				5	0.446	0.888	32.4544	1.0000
				10	0.446	0.888	32.4538	1.0000
				∞	0.446	0.888	32.4536	1
	.01	1.7268	-0.5768	2	0.002	0.999	62.5420	1.0000
				5	0.002	0.999	62.5420	1.0000
				10	0.002	0.999	62.5420	1.0000
				∞	0.002	0.999	62.5420	1
0.01	.9	1.1500	-1.3585	2	0.450	0.845	2.6444	1.0481
				5	0.462	0.854	2.5460	1.0091
				10	0.464	0.855	2.5286	1.0022
				∞	0.466	0.855	2.5231	1
	.5	1.1500	-1.0584	2	0.366	0.898	4.1680	1.0034
				5	0.368	0.899	4.1575	1.0008
				10	0.368	0.899	4.1548	1.0002
				∞	0.368	0.899	4.1540	1
	.1	1.1500	-0.5874	2	0.002	0.999	6.2538	1.0000
				5	0.002	0.999	6.2537	1.0000
				10	0.002	0.999	6.2537	1.0000
				∞	0.002	0.999	6.2537	1
0.1	.9	0.5626	-0.7711	2	0.030	0.988	0.6241	1.0062
				5	0.052	0.981	0.6212	1.0015
				10	0.056	0.980	0.6205	1.0004
				∞	0.058	0.979	0.6203	1
	.7	0.5626	-0.6090	2	0.002	0.999	0.6261	1.0008
				5	0.002	0.999	0.6257	1.0002
				10	0.002	0.999	0.6256	1.0001
				∞	0.002	0.999	0.6256	1

TABLE 6. Sensitivities of avar($\hat{\mu}_0$) When $\delta = 0.5$

P_u	P_h						
0.0001	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7	0.8	0.9	0.95	0.99
		0.00001	1.1199	1.1091	1.0813	1.0603	1.0339
		0.00003	1.0454	1.0388	1.0265	1.0147	1.0204
		0.00005	1.0210	1.0162	1.0084	1.0045	1.0248
		0.0001	1.0046	1.0023	1	1.0031	1.0448
		0.0002	1.0111	1.0090	1.0106	1.0209	1.0863
		0.0003	1.0295	1.0246	1.0284	1.0430	1.1232
		0.0005	1.0723	1.0654	1.0677	1.0881	1.1891
	0.1	$\tilde{P}_u \backslash \tilde{P}_h$	0.06	0.08	0.1	0.12	0.14
		0.00001	1.0826	1.1084	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0520	1.0599
		0.00005	1.0013	1.0072	1.0146	1.0226	1.0299
		0.0001	1.0099	1.0016	1	1.0012	1.0031
		0.0002	1.0644	1.0340	1.0178	1.0101	1.0046
		0.0003	1.1297	1.0814	1.0524	1.0342	1.0234
		0.0005	1.2670	1.1790	1.1301	1.0967	1.0718
	0.01	$\tilde{P}_u \backslash \tilde{P}_h$	0.006	0.008	0.01	0.012	0.014
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
		0.00003	1.0313	1.0579	1.0843	1.1093	1.1301
		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.0001	1.0232	1.0042	1	1.0024	1.0076
		0.0002	1.1744	1.0869	1.0465	1.0232	1.0110
		0.0003	1.3941	1.2217	1.1359	1.0869	1.0560
		0.0005	1.8870	1.5967	1.3890	1.2748	1.1993
	0.001	$\tilde{P}_u \backslash \tilde{P}_h$	0.0006	0.0008	0.001	0.0012	0.0014
		0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
		0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.0001	1.0000	1.0000	1	1.0000	1.0119
		0.0002	1.0002	1.0001	1.0001	1.0000	1.0000
		0.0003	1.0004	1.0003	1.0002	1.0001	1.0001
		0.0005	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7	0.8	0.9	0.95	0.99
		0.0001	1.1637	1.1588	1.1344	1.1092	1.0634
		0.0003	1.0519	1.0519	1.0423	1.0310	1.0233
		0.0005	1.0202	1.0196	1.0153	1.0099	1.0192
		0.001	1.0036	1.0015	1	1.0023	1.0342
		0.002	1.0288	1.0207	1.0194	1.0275	1.0814
		0.003	1.0742	1.0585	1.0537	1.0622	1.1300
		0.005	1.1756	1.1458	1.1286	1.1398	1.2274

Table 6. contd...

0.5

$\tilde{P}_u \backslash \tilde{P}_h$	0.3	0.4	0.5	0.6	0.7
0.0001	1.1067	1.1324	1.1540	1.1640	1.1726
0.0003	1.0196	1.0346	1.0476	1.0591	1.0643
0.0005	1.0021	1.0090	1.0174	1.0243	1.0299
0.001	1.0121	1.0019	1	1.0012	1.0036
0.002	1.0839	1.0435	1.0229	1.0127	1.0071
0.003	1.1711	1.1032	1.0641	1.0415	1.0300
0.005	1.3720	1.2365	1.1648	1.1189	1.0884

0.1

$\tilde{P}_u \backslash \tilde{P}_h$	0.06	0.08	0.1	0.12	0.14
0.0001	1.1676	1.2187	1.2588	1.2957	1.3208
0.0003	1.0304	1.0567	1.0829	1.1075	1.1279
0.0005	1.0025	1.0151	1.0302	1.0465	1.0639
0.001	1.0235	1.0043	1	1.0024	1.0077
0.002	1.1742	1.0883	1.0455	1.0224	1.0104
0.003	1.3917	1.2221	1.1355	1.0861	1.0552
0.005	1.9168	1.5895	1.3866	1.2676	1.1925

0.01

$\tilde{P}_u \backslash \tilde{P}_h$	0.006	0.008	0.01	0.012	0.014
0.0001	1.5741	1.7417	1.8847	2.0144	2.1219
0.0003	1.0931	1.1904	1.2813	1.3641	1.4409
0.0005	1.0000	1.0370	1.0951	1.1523	1.2099
0.001	1.0000	1.0000	1	1.0000	1.0118
0.002	1.0002	1.0001	1.0001	1.0000	1.0000
0.003	1.0004	1.0003	1.0002	1.0001	1.0001
0.005	1.0008	1.0006	1.0004	1.0003	1.0003

0.01 0.9

$\tilde{P}_u \backslash \tilde{P}_h$	0.7	0.8	0.9	0.95	0.99
0.001	1.2622	1.2707	1.2558	1.2300	1.1593
0.003	1.0712	1.0809	1.0806	1.0727	1.0519
0.005	1.0209	1.0269	1.0300	1.0266	1.0235
0.01	1.0077	1.0018	1	1.0014	1.0201
0.02	1.1039	1.0658	1.0421	1.0396	1.0718
0.03	1.2601	1.1757	1.1245	1.1118	1.1453
0.05	1.6885	1.4829	1.3411	1.2943	1.3125

0.5

$\tilde{P}_u \backslash \tilde{P}_h$	0.3	0.4	0.5	0.6	0.7
0.001	1.2191	1.2838	1.3305	1.3719	1.3984
0.003	1.0379	1.0733	1.1084	1.1397	1.1621
0.005	1.0021	1.0184	1.0393	1.0616	1.0835
0.01	1.0403	1.0069	1	1.0040	1.0129
0.02	1.2905	1.1338	1.0622	1.0266	1.0094
0.03	1.4966	1.3468	1.1921	1.1070	1.0591
0.05	1.4968	1.4967	1.4966	1.3568	1.2252

0.1

$\tilde{P}_u \backslash \tilde{P}_h$	0.06	0.08	0.1	0.12	0.14
0.001	1.5684	1.7352	1.8773	2.0058	2.1119
0.003	1.0945	1.1894	1.2803	1.3631	1.4397
0.005	1.0000	1.0396	1.0964	1.1564	1.2150
0.01	1.0000	1.0000	1	1.0000	1.0171

Table 6. contd...

		0.02	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05	1.0008	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7	0.8	0.9	0.95	0.99
		0.01	1.6612	1.7413	1.7804	1.7671	1.6443
		0.03	1.1437	1.2040	1.2552	1.2711	1.2480
		0.05	1.0210	1.0537	1.0953	1.1154	1.1178
		0.1	1.0036	1.0036	1	1.0037	1.0122
		0.15	1.0036	1.0036	1.0036	1.0036	1.0037
		0.2	1.0037	1.0037	1.0037	1.0037	1.0038
		0.3	1.0038	1.0038	1.0038	1.0038	1.0041
	0.7	$\tilde{P}_u \setminus \tilde{P}_h$	0.5	0.6	0.7	0.8	0.9
		0.01	1.7260	1.8678	1.9920	2.0973	2.1546
		0.03	1.1225	1.2148	1.3064	1.3962	1.4723
		0.05	1.0000	1.0350	1.0970	1.1643	1.2377
		0.10	1.0000	1.0000	1	1.0000	1.0224
		0.15	1.0001	1.0001	1.0000	1.0000	1.0000
		0.20	1.0002	1.0001	1.0001	1.0001	1.0001
		0.30	1.0004	1.0003	1.0002	1.0002	1.0002

TABLE 7. Sensitivities Of avar($\hat{\mu}_0$) When δ = 0.9

P_u	P_h						
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.1083	1.0997	1.0811	1.0619	1.0377
		0.00003	1.0386	1.0336	1.0237	1.0174	1.0132
		0.00005	1.0165	1.0146	1.0084	1.0055	1.0125
		0.00010	1.0031	1.0014	1	1.0020	1.0239
		0.00020	1.0127	1.0091	1.0107	1.0171	1.0556
		0.00030	1.0300	1.0279	1.0284	1.0376	1.0862
		0.00050	1.0737	1.0666	1.0677	1.0805	1.1381
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0826	1.1084	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0520	1.0599
		0.00005	1.0013	1.0072	1.0146	1.0226	1.0299
		0.00010	1.0099	1.0016	1	1.0012	1.0031
		0.00020	1.0644	1.0340	1.0178	1.0101	1.0046
		0.00030	1.1298	1.0814	1.0524	1.0342	1.0234
		0.00050	1.2671	1.1790	1.1301	1.0968	1.0718
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
		0.00003	1.0313	1.0579	1.0843	1.1093	1.1301

Table 7. contd...

		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.00010	1.0232	1.0042	1	1.0024	1.0076
		0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
		0.00030	1.3941	1.2217	1.1359	1.0869	1.0560
		0.00050	1.8870	1.5967	1.3890	1.2748	1.1993
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
		0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.1497	1.1469	1.1325	1.1101	1.0759
		0.00030	1.0473	1.0458	1.0413	1.0337	1.0220
		0.00050	1.0172	1.0175	1.0147	1.0110	1.0126
		0.00100	1.0031	1.0010	1	1.0013	1.0175
		0.00200	1.0322	1.0218	1.0183	1.0226	1.0536
		0.00300	1.0765	1.0604	1.0517	1.0549	1.0919
		0.00500	1.1851	1.1489	1.1254	1.1288	1.1724
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1060	1.1317	1.1534	1.1637	1.1727
		0.00030	1.0193	1.0342	1.0473	1.0589	1.0677
		0.00050	1.0020	1.0088	1.0172	1.0242	1.0321
		0.00100	1.0123	1.0020	1	1.0012	1.0035
		0.00200	1.0844	1.0438	1.0231	1.0127	1.0069
		0.00300	1.1717	1.1036	1.0644	1.0416	1.0281
		0.00500	1.3729	1.2372	1.1653	1.1191	1.0883
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1676	1.2187	1.2588	1.2956	1.3208
		0.00030	1.0304	1.0567	1.0829	1.1075	1.1279
		0.00050	1.0025	1.0151	1.0302	1.0465	1.0639
		0.00100	1.0235	1.0043	1	1.0024	1.0077
		0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
		0.00300	1.3917	1.2221	1.1355	1.0861	1.0552
		0.00500	1.9169	1.5896	1.3866	1.2676	1.1925
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5741	1.7417	1.8847	2.0144	2.1219
		0.00030	1.0931	1.1904	1.2813	1.3641	1.4409
		0.00050	1.0000	1.0370	1.0951	1.1523	1.2099
		0.00100	1.0000	1.0000	1	1.0000	1.0118
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

Table 7. contd...

		$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01	0.9	0.00100	1.2431	1.2597	1.2500	1.2348	1.1841
		0.00300	1.0651	1.0757	1.0808	1.0777	1.0616
		0.00500	1.0167	1.0241	1.0302	1.0291	1.0261
		0.01000	1.0089	1.0018	1	1.0009	1.0102
		0.02000	1.1107	1.0686	1.0418	1.0345	1.0475
		0.03000	1.2709	1.1803	1.1209	1.1035	1.1091
		0.05000	1.7065	1.4833	1.3347	1.2763	1.2565

		$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00100	1.2179	1.2826	1.3294	1.3712	1.3985
		0.00300	1.0375	1.0728	1.1079	1.1393	1.1661
		0.00500	1.0021	1.0181	1.0390	1.0613	1.0834
		0.01000	1.0405	1.0071	1	1.0039	1.0128
		0.02000	1.2912	1.1343	1.0625	1.0267	1.0094
		0.03000	1.4975	1.3475	1.1894	1.1073	1.0591
		0.05000	1.4977	1.4976	1.4975	1.3575	1.2255

		$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00100	1.5683	1.7352	1.8772	2.0058	2.1119
		0.00300	1.0945	1.1894	1.2802	1.3630	1.4397
		0.00500	1.0000	1.0396	1.0964	1.1564	1.2150
		0.01000	1.0000	1.0000	1	1.0000	1.0171
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0008	1.0006	1.0004	1.0003	1.0003

		$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.1	0.9	0.01000	1.6214	1.7100	1.7654	1.7706	1.7120
		0.03000	1.1314	1.1928	1.2531	1.2763	1.2801
		0.05000	1.0173	1.0507	1.0943	1.1205	1.1377
		0.10000	1.0049	1.0049	1	1.0043	1.0181
		0.15000	1.0049	1.0049	1.0049	1.0049	1.0030
		0.20000	1.0050	1.0050	1.0049	1.0049	1.0050
		0.30000	1.0051	1.0051	1.0051	1.0051	1.0052

		$\tilde{P}_u \setminus \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
	0.7	0.01000	1.7168	1.8578	1.9825	2.1019	2.1796
		0.03000	1.1232	1.2121	1.3031	1.3983	1.4892
		0.05000	1.0000	1.0344	1.0957	1.1686	1.2478
		0.10000	1.0000	1.0000	1	1.0000	1.0260
		0.15000	1.0001	1.0001	1.0000	1.0000	1.0000
		0.20000	1.0002	1.0001	1.0001	1.0001	1.0001
		0.30000	1.0004	1.0003	1.0002	1.0002	1.0002

TABLE 8. Sensitivities Of avar($\hat{\mu}_0$) When δ = 1.0

P_u	P_h	$\tilde{P}_u \setminus \tilde{P}_h$					
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.1069	1.0986	1.0805	1.0666	1.0366
		0.00003	1.0378	1.0330	1.0261	1.0172	1.0132
		0.00005	1.0160	1.0142	1.0082	1.0054	1.0112
		0.00010	1.0030	1.0013	1	1.0016	1.0208
		0.00020	1.0129	1.0093	1.0109	1.0170	1.0505
		0.00030	1.0304	1.0283	1.0288	1.0376	1.0798
		0.00050	1.0745	1.0673	1.0682	1.0805	1.1301
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0826	1.1084	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0520	1.0599
		0.00005	1.0013	1.0072	1.0146	1.0226	1.0299
		0.00010	1.0099	1.0016	1	1.0012	1.0031
		0.00020	1.0644	1.0340	1.0178	1.0101	1.0046
		0.00030	1.1298	1.0814	1.0524	1.0342	1.0234
		0.00050	1.2671	1.1790	1.1301	1.0968	1.0718
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
		0.00003	1.0313	1.0579	1.0843	1.1093	1.1301
		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.00010	1.0232	1.0042	1	1.0024	1.0076
		0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
		0.00030	1.3941	1.2217	1.1359	1.0869	1.0560
		0.00050	1.8870	1.5967	1.3890	1.2748	1.1993
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
		0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.1480	1.1455	1.1316	1.1151	1.0797
		0.00030	1.0464	1.0450	1.0409	1.0334	1.0228
		0.00050	1.0167	1.0170	1.0144	1.0108	1.0122
		0.00100	1.0030	1.0009	1	1.0011	1.0152
		0.00200	1.0326	1.0221	1.0185	1.0225	1.0493
		0.00300	1.0773	1.0610	1.0522	1.0549	1.0864
		0.00500	1.1863	1.1499	1.1261	1.1290	1.1651

Table 8. contd...

		$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00010	1.1059	1.1316	1.1533	1.1637	1.1728
		0.00030	1.0193	1.0342	1.0473	1.0589	1.0677
		0.00050	1.0020	1.0088	1.0172	1.0242	1.0322
		0.00100	1.0123	1.0020	1	1.0012	1.0035
		0.00200	1.0844	1.0438	1.0231	1.0127	1.0069
		0.00300	1.1718	1.1036	1.0644	1.0416	1.0281
		0.00500	1.3729	1.2372	1.1653	1.1191	1.0882

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00010	1.1676	1.2187	1.2588	1.2956	1.3208
		0.00030	1.0304	1.0567	1.0829	1.1075	1.1279
		0.00050	1.0025	1.0151	1.0302	1.0465	1.0639
		0.00100	1.0235	1.0043	1	1.0024	1.0077
		0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
		0.00300	1.3917	1.2221	1.1355	1.0861	1.0552
		0.00500	1.9169	1.5896	1.3866	1.2676	1.1925

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
	0.01	0.00010	1.5741	1.7417	1.8847	2.0144	2.1219
		0.00030	1.0931	1.1904	1.2813	1.3641	1.4409
		0.00050	1.0000	1.0370	1.0951	1.1523	1.2099
		0.00100	1.0000	1.0000	1	1.0000	1.0118
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

		$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01	0.9	0.00100	1.2408	1.2576	1.2549	1.2341	1.1894
		0.00300	1.0640	1.0747	1.0801	1.0773	1.0635
		0.00500	1.0162	1.0236	1.0298	1.0288	1.0266
		0.01000	1.0082	1.0019	1	1.0008	1.0089
		0.02000	1.1116	1.0693	1.0423	1.0346	1.0444
		0.03000	1.2672	1.1813	1.1216	1.1010	1.1024
		0.05000	1.6998	1.4850	1.3359	1.2721	1.2465

		$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00100	1.2178	1.2825	1.3293	1.3712	1.3986
		0.00300	1.0375	1.0728	1.1079	1.1393	1.1661
		0.00500	1.0021	1.0181	1.0390	1.0613	1.0834
		0.01000	1.0406	1.0071	1	1.0039	1.0138
		0.02000	1.2913	1.1343	1.0625	1.0267	1.0094
		0.03000	1.4975	1.3476	1.1894	1.1073	1.0574
		0.05000	1.4978	1.4976	1.4975	1.3576	1.2221

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00100	1.5683	1.7352	1.8772	2.0058	2.1119
		0.00300	1.0945	1.1894	1.2802	1.3630	1.4397
		0.00500	1.0000	1.0396	1.0964	1.1564	1.2149
		0.01000	1.0000	1.0000	1	1.0000	1.0171

Table 8. contd.....

P_u	P_h						
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0008	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	1.6246	1.7052	1.7615	1.7681	1.7214
		0.03000	1.1325	1.1911	1.2514	1.2794	1.2885
		0.05000	1.0177	1.0500	1.0935	1.1223	1.1424
		0.10000	1.0050	1.0050	1	1.0045	1.0192
		0.15000	1.0051	1.0051	1.0050	1.0050	1.0026
		0.20000	1.0051	1.0051	1.0051	1.0051	1.0052
		0.30000	1.0053	1.0053	1.0052	1.0052	1.0053
	0.7	$\tilde{P}_u \setminus \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.7160	1.8570	1.9922	2.1014	2.1802
		0.03000	1.1230	1.2118	1.3071	1.3981	1.4893
		0.05000	1.0000	1.0344	1.0978	1.1685	1.2478
		0.10000	1.0000	1.0000	1	1.0000	1.0260
		0.15000	1.0001	1.0001	1.0000	1.0000	1.0000
		0.20000	1.0002	1.0001	1.0001	1.0001	1.0001
		0.30000	1.0004	1.0003	1.0002	1.0002	1.0002

TABLE 9. Sensitivitties Of avar($\hat{\mu}_0$) When δ = 3.0

P_u	P_h						
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.1181	1.1063	1.0839	1.0635	1.0382
		0.00003	1.0445	1.0374	1.0251	1.0159	1.0133
		0.00005	1.0224	1.0172	1.0092	1.0049	1.0099
		0.00010	1.0049	1.0022	1	1.0026	1.0200
		0.00020	1.0103	1.0082	1.0115	1.0189	1.0502
		0.00030	1.0281	1.0259	1.0298	1.0403	1.0765
		0.00050	1.0700	1.0634	1.0699	1.0843	1.1307
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0826	1.1085	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0521	1.0599
		0.00005	1.0013	1.0072	1.0147	1.0226	1.0299
		0.00010	1.0099	1.0016	1	1.0012	1.0031
		0.00020	1.0643	1.0340	1.0194	1.0101	1.0046
		0.00030	1.1297	1.0813	1.0524	1.0342	1.0234
		0.00050	1.2670	1.1790	1.1301	1.0967	1.0718
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
		0.00003	1.0313	1.0579	1.0843	1.1093	1.1301

Table 9. contd...

		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.00010	1.0232	1.0042	1	1.0024	1.0076
		0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
		0.00030	1.3941	1.2217	1.1359	1.0869	1.0560
		0.00050	1.8870	1.5967	1.3890	1.2748	1.1993
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
		0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.1629	1.1569	1.1325	1.1087	1.0796
		0.00030	1.0544	1.0510	1.0413	1.0331	1.0245
		0.00050	1.0216	1.0206	1.0147	1.0108	1.0114
		0.00100	1.0036	1.0016	1	1.0017	1.0141
		0.00200	1.0296	1.0214	1.0184	1.0253	1.0480
		0.00300	1.0718	1.0565	1.0520	1.0591	1.0851
		0.00500	1.1774	1.1427	1.1304	1.1353	1.1637
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1079	1.1336	1.1495	1.1647	1.1668
		0.00030	1.0201	1.0351	1.0481	1.0563	1.0645
		0.00050	1.0023	1.0093	1.0177	1.0246	1.0301
		0.00100	1.0118	1.0018	1	1.0013	1.0037
		0.00200	1.0832	1.0430	1.0226	1.0126	1.0080
		0.00300	1.1701	1.1025	1.0636	1.0434	1.0300
		0.00500	1.3705	1.2354	1.1641	1.1185	1.0915
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1677	1.2187	1.2589	1.2957	1.3208
		0.00030	1.0304	1.0568	1.0829	1.1075	1.1279
		0.00050	1.0025	1.0151	1.0302	1.0465	1.0639
		0.00100	1.0234	1.0043	1	1.0024	1.0077
		0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
		0.00300	1.3917	1.2221	1.1355	1.0861	1.0552
		0.00500	1.9168	1.5895	1.3866	1.2676	1.1960
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5741	1.7417	1.8847	2.0144	2.1219
		0.00030	1.0931	1.1904	1.2813	1.3641	1.4409
		0.00050	1.0000	1.0370	1.0951	1.1523	1.2099
		0.00100	1.0000	1.0000	1	1.0000	1.0118
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

Table 9. contd...

0.01	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00100	1.2694	1.2706	1.2555	1.2319	1.1953
		0.00300	1.0743	1.0837	1.0834	1.0764	1.0661
		0.00500	1.0213	1.0283	1.0299	1.0285	1.0278
		0.01000	1.0079	1.0018	1	1.0011	1.0081
		0.02000	1.1042	1.0661	1.0424	1.0373	1.0422
		0.03000	1.2606	1.1803	1.1250	1.1082	1.1016
		0.05000	1.6896	1.4840	1.3420	1.2888	1.2502
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00100	1.2210	1.2859	1.3324	1.3734	1.3993
		0.00300	1.0385	1.0741	1.1093	1.1368	1.1626
		0.00500	1.0023	1.0188	1.0398	1.0621	1.0813
		0.01000	1.0398	1.0067	1	1.0041	1.0122
		0.02000	1.2894	1.1330	1.0617	1.0275	1.0101
		0.03000	1.4951	1.3500	1.1912	1.1088	1.0606
		0.05000	1.4953	1.4952	1.4951	1.3602	1.2279
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00100	1.5685	1.7354	1.8774	2.0059	2.1120
		0.00300	1.0945	1.1895	1.2803	1.3631	1.4398
		0.00500	1.0000	1.0397	1.0964	1.1564	1.2118
		0.01000	1.0000	1.0000	1	1.0000	1.0171
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0008	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	1.6699	1.7490	1.7884	1.7897	1.7488
		0.03000	1.1462	1.2031	1.2583	1.2804	1.3012
		0.05000	1.0216	1.0547	1.0968	1.1204	1.1470
		0.10000	1.0030	1.0030	1	1.0043	1.0193
		0.15000	1.0031	1.0031	1.0031	1.0031	1.0020
		0.20000	1.0031	1.0031	1.0031	1.0031	1.0032
		0.30000	1.0033	1.0033	1.0033	1.0033	1.0033
	0.7	$\tilde{P}_u \setminus \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.7290	1.8798	1.9935	2.0971	2.1545
		0.03000	1.1247	1.2147	1.3062	1.3905	1.4719
		0.05000	1.0000	1.0341	1.0962	1.1635	1.2371
		0.10000	1.0000	1.0000	1	1.0000	1.0202
		0.15000	1.0001	1.0001	1.0000	1.0000	1.0000
		0.20000	1.0002	1.0001	1.0001	1.0001	1.0001
		0.30000	1.0004	1.0003	1.0002	1.0002	1.0002

Table 10. Sensitivities Of $avar(\hat{\mu}0)$ When δ = 4.0

P_u	P_h	$\tilde{P}_u \setminus \tilde{P}_h$					
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.1315	1.1169	1.0846	1.0575	1.0324
		0.00003	1.0525	1.0434	1.0254	1.0136	1.0165
		0.00005	1.0260	1.0193	1.0094	1.0042	1.0198
		0.00010	1.0070	1.0034	1	1.0042	1.0369
		0.00020	1.0108	1.0081	1.0113	1.0229	1.0750
		0.00030	1.0248	1.0225	1.0295	1.0488	1.1137
		0.00050	1.0639	1.0618	1.0695	1.0965	1.1727
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0827	1.1085	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0521	1.0599
		0.00005	1.0013	1.0072	1.0147	1.0226	1.0299
		0.00010	1.0099	1.0016	1	1.0012	1.0031
		0.00020	1.0643	1.0340	1.0194	1.0101	1.0046
		0.00030	1.1297	1.0813	1.0524	1.0342	1.0234
		0.00050	1.2670	1.1789	1.1301	1.0967	1.0718
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3236
		0.00003	1.0313	1.0579	1.0843	1.1093	1.1301
		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.00010	1.0232	1.0042	1	1.0024	1.0076
		0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
		0.00030	1.3941	1.2217	1.1359	1.0869	1.0560
		0.00050	1.8870	1.5967	1.3890	1.2748	1.1993
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5754	1.7432	1.8864	2.0163	2.1241
		0.00003	1.0936	1.1912	1.2785	1.3652	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.1790	1.1703	1.1350	1.1032	1.0696
		0.00030	1.0632	1.0581	1.0426	1.0283	1.0215
		0.00050	1.0271	1.0233	1.0154	1.0086	1.0165
		0.00100	1.0050	1.0023	1	1.0029	1.0283
		0.00200	1.0262	1.0184	1.0193	1.0304	1.0730
		0.00300	1.0693	1.0543	1.0536	1.0666	1.1200
		0.00500	1.1674	1.1390	1.1331	1.1463	1.2103

Table 10. contd...

		$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00010	1.1088	1.1345	1.1502	1.1651	1.1667
		0.00030	1.0204	1.0355	1.0485	1.0566	1.0645
		0.00050	1.0024	1.0094	1.0179	1.0247	1.0302
		0.00100	1.0116	1.0018	1	1.0009	1.0031
		0.00200	1.0827	1.0427	1.0224	1.0125	1.0081
		0.00300	1.1694	1.1019	1.0633	1.0433	1.0320
		0.00500	1.3694	1.2347	1.1636	1.1183	1.0917

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00010	1.1677	1.2188	1.2589	1.2957	1.3209
		0.00030	1.0304	1.0568	1.0829	1.1075	1.1280
		0.00050	1.0025	1.0151	1.0302	1.0465	1.0640
		0.00100	1.0234	1.0043	1	1.0024	1.0077
		0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
		0.00300	1.3916	1.2221	1.1354	1.0861	1.0552
		0.00500	1.9167	1.5895	1.3865	1.2675	1.1960

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
	0.01	0.00010	1.5741	1.7417	1.8847	2.0144	2.1219
		0.00030	1.0931	1.1904	1.2813	1.3641	1.4409
		0.00050	1.0000	1.0370	1.0951	1.1523	1.2099
		0.00100	1.0000	1.0000	1	1.0000	1.0118
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

		$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01	0.9	0.00100	1.2853	1.2900	1.2624	1.2280	1.1690
		0.00300	1.0818	1.0900	1.0837	1.0719	1.0545
		0.00500	1.0266	1.0319	1.0301	1.0263	1.0248
		0.01000	1.0070	1.0019	1	1.0017	1.0161
		0.02000	1.1001	1.0629	1.0442	1.0424	1.0641
		0.03000	1.2540	1.1750	1.1282	1.1164	1.1353
		0.05000	1.6790	1.4755	1.3475	1.3064	1.2990

		$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00100	1.2224	1.2873	1.3336	1.3672	1.3917
		0.00300	1.0390	1.0747	1.1099	1.1373	1.1587
		0.00500	1.0024	1.0190	1.0401	1.0624	1.0814
		0.01000	1.0395	1.0066	1	1.0037	1.0123
		0.02000	1.2886	1.1325	1.0614	1.0273	1.0108
		0.03000	1.4941	1.3492	1.1906	1.1085	1.0622
		0.05000	1.4943	1.4942	1.4941	1.3594	1.2310

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00100	1.5685	1.7354	1.8775	2.0060	2.1120
		0.00300	1.0945	1.1895	1.2803	1.3631	1.4398
		0.00500	1.0000	1.0397	1.0964	1.1564	1.2118
		0.01000	1.0000	1.0000	1	1.0000	1.0171

Table 10. contd...

		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0008	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	1.7161	1.7855	1.8069	1.7790	1.6895
		0.03000	1.1577	1.2161	1.2618	1.2765	1.2677
		0.05000	1.0251	1.0602	1.0985	1.1160	1.1288
		0.10000	1.0019	1.0019	1	1.0031	1.0134
		0.15000	1.0020	1.0020	1.0020	1.0020	1.0021
		0.20000	1.0020	1.0020	1.0020	1.0021	1.0022
		0.30000	1.0022	1.0022	1.0022	1.0022	1.0024
	0.7	$\tilde{P}_u \backslash \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.7392	1.8814	2.0040	2.0934	2.1309
		0.03000	1.1267	1.2177	1.3055	1.3887	1.4559
		0.05000	1.0000	1.0347	1.0953	1.1622	1.2276
		0.10000	1.0000	1.0000	1	1.0000	1.0155
		0.15000	1.0001	1.0001	1.0000	1.0000	1.0000
		0.20000	1.0002	1.0001	1.0001	1.0001	1.0001
		0.30000	1.0004	1.0003	1.0002	1.0002	1.0002

APPENDIX

ALGORITHM OF COMPUTATIONAL EXPERIMENTS

The two-step optimization technique is adopted to our problem according to the following steps:

1. Provide guesstimates of P_u and P_h and δ and the ranges of their plausible values. Calculate β_0 and β_1 according to (4.6.5) and (4.6.6), respectively.
2. Periodic inspection times at stress level i is calculated by
   $t_{ij} = j/k$, for $i = 1, 2$ and $j = 1, 2, \ldots, K$.
3. Set $s_0 = 0$, $s_2 = 1$ and generate 499 values of s_1 between (0, 1) at an interval of 0.002. Calculate θ_i according to (4.2.2), for $i = 1, 2$.
4. For each value of t_{ij} and θ_i and for given δ. Calculate Q_i according to (4.4.6).
5. For each value of s_1 and Q_i. Calculate α_1 according to (4.5.3) and finally asymptotic variance of $\hat{\mu}_0$ is calculated according to (4.5.2).
6. Select a value of s_1 and corresponding value of α_1 such that asymptotic variance of $\hat{\mu}_0$ is minimized among all values of s_1. Otherwise, select next s_1. If a convergence criterion is met, then stop.

CHAPTER 5

OPTIMAL ACCELERATED LIFE TEST DESIGNS FOR THE LOG-LOGISTIC AND BURR TYPE XII DISTRIBUTIONS UNDER PERIODIC INSPECTION AND TYPE-I CENSORING

5.1 INTRODUCTION

Most of the previous works on the optimal designs of ALTs assume that the lifetime distribution of a test unit is either exponential, Weibull (i.e., Bai and Chun(1991); Bai and Chung(1992); Bai, Kim and Lee(1989); Chernoff(1962); Chernoff(1953); DeGroot and Goel(1979); Ehrenfeld(1962); Meeker(1984); Miller and Nelson(1983); Nelson and Meeker(1978); Seo and Yum(1991); Yum and Choi(1989)) or Normal, log-Normal (i.e., Kielpinski and Nelson(1975); Meeker(1984); Nelson and Kielpinski(1976)).

This study considers Burr Type XII distribution (1942), as the lifetime distribution of the test items. It has log-logistic distribution as a special case for $m = 1$. It has the advantage of having simple algebraic formulations for the reliability and hazard rate functions, like the Weibull and exponential distributions. It is therefore more convenient in handling censored data than the log-normal distribution while providing a good approximation except in the extreme tails. Also the hazard function of log-logistic distribution is identical to the Weibull's hazard function aside from the denominator factor. It is monotone decreasing from ∞ if $\delta < 1$ and is monotone decreasing from θ if $\delta = 1$. For $\delta > 1$, it's hazard rate function resembles hazard function of log-normal distribution.

The present investigation is an attempt to combine such important feature of life tests as acceleration & periodic inspection. An asymptotically optimal ALT plan for the Burr type XII distribution is developed under the assumptions of Type I censoring and periodic inspection at two overstress levels. The use test stress and high test stress are specified while low test stress is optimally determined and proportion of test items are allocated to each stress. This investigation extends the Yum and Choi (1989) or Seo and Yum (1991) work in that a new more flexible life time model, Burr Type XII, is assumed to describe the failure mechanism of test units and a new software has been developed to carry out the computation which considerably reduces the time of work to get the results. It is assumed that a log-linear relationship exists between the Burr scale parameter and the stress, and that Burr shape parameter is constant and is independent of the stress. ML estimation method is used to estimate the unknown parameters in this relationship. Then, the asymppototic properties of those ML estimators are used to approximate the variance of the estimated mean at the use condition. The asymptotic variance (AsVar) of the estimated mean or q^{th} quantile of the lifetime distribution at use or design stress is adopted as an optimality criterion.

Computational experiments are conducted for various combinations of pararmeters involved to examine how optimal plans vary with respect to these parameters. Sensitivity analysis is also conducted for various combinations of parameters to assess the effect of inaccuracy in the "guesstimates" of the unknown parameters on the optimal plan.

5.2 THE MODEL

Basic Assumptions

1. ℓ stress levels s_1, s_2, ..., s_ℓ are considered, where $s_1 < s_2 < \ldots < s_\ell$ may denote high temperature or voltage etc. This work deals with the case of three stress levels (ℓ = 2) in

which s_0 denotes use stress level, s_1 the low stress level and s_2 the high stress level, such that $s_0 < s_1 < s_2$.

2. At any level of stress s, the lifetimes (T) of test items follow the Burr Type XII distribution with shape parameters m, δ and scale parameter θ. That is,

$$f(t) = m(\delta/\theta)\left(t/\theta\right)^{\delta-1}\left(1 + (t/\theta)^{\delta}\right)^{-(m+1)} , \quad t \geq 0,\ m,\delta,\theta > 0 . \tag{5.2.1}$$

For $m = 1$, the distribution $f(t)$ becomes a log-logistic distribution with shape parameter δ and scale parameter θ. That is,

$$f(t) = (\delta/\theta)\left(t/\theta\right)^{\delta-1}\left(1 + (t/\theta)^{\delta}\right)^{-2} , \quad t \geq 0,\ \delta,\theta > 0 . \tag{5.2.2}$$

For $m = 2$, the lifetime distribution becomes

$$f(t) = 2(\delta/\theta)\left(t/\theta\right)^{\delta-1}\left(1 + (t/\theta)^{\delta}\right)^{-3} , \quad t \geq 0,\ \delta,\theta > 0 . \tag{5.2.3}$$

and so on.

3. The mean lifetime (θ) and the stress level s are exponentially related as

$$\theta = e^{\beta_0 + \beta_1 s} , \tag{5.2.4}$$

where β_0 and β_1 are unknown parameters depending on the nature of the product and the test method (i.e., see Meeker(1986); Miller and Nelson(1983); Nelson and Kielpinski(1976)).

4. δ and m are known parameters, and are indepedent of stress.
5. The lifetimes of test units at stress level s_i are independent and identically distributed.
6. At use (Design) condition, t_q is the q^{th} quantile of the lifetime distribution.

Test Procedure

1. The use test stress (s_0) and high test stress (s_2) are prespecified, while the test stress (s_1) is to be optimally determined.
2. Test units (n_i) allocated to s_i out of total N test units is

$$n_i = \alpha_i N, \quad \sum_{i=1}^{\ell} \alpha_i = 1, \ \alpha_i > 0, \ i = 1, 2, \ldots, \ell . \qquad (5.2.5)$$

For $\ell = 2$

$$n_1 = \alpha_1 N, \ n_2 = \alpha_2 N = (1 - \alpha_1) N , \qquad (5.2.6)$$

where α_i is to be optimally determined.

3. The test items (n_i) are initially placed on life test at stress level s_i and run until a prespecified time t_{ci} (i.e., Type I censoring is assumed).
4. At specified point in time t_{i1}, t_{i2}, ..., $t_{i,K(i)}$, the inspections are carried out, where $t_{i,K(i)} = t_{ci}$, $t_{io} = 0$ and $t_{i,K(i)+1} = \infty$. Let the number of test items failed during (t_{ij-1}, t_{ij}) at stress level s_i is x_{ij} and the probability of failure during (t_{ij-1}, t_{ij}) is P_{ij}, j = 1, 2, ..., K(i)+1.

The grouped data $\{ x_{ij}$, i = 1, 2, ..., ℓ; j = 1, 2, ..., $K(i)+1\}$ are used to estimate β_0 and β_1. The estimated relationship is then extrapolated to estimate mean lifetime at the use condition. Logarithm of the mean lifetime θ at use condition is defined by

$$\mu_0 = \ln\theta_0 = \beta_0 + \beta_1 s_0 . \qquad (5.2.7)$$

Let t_q be the q^{th} quantile of the lifetime distribution at use condition. We want to estimate

$$\ln(t_q) = \beta_0 + \beta_1 s_0 + 1/\delta . \ln\left\{\frac{1-(1-q)^{1/m}}{(1-q)^{1/m}}\right\}$$

$$= y_q \text{ (say)} \quad . \tag{5.2.8}$$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the ML estimates of β_0 and β_1, respectively. Then, y_q is estimated as

$$\hat{y}_q = \hat{\mu} + 1/\delta . \ln\left\{\frac{1-(1-q)^{1/m}}{(1-q)^{1/m}}\right\} , \tag{5.2.9}$$

where

$$\hat{\mu} = \hat{\beta}_0 + \hat{\beta}_1 s_0 \quad . \tag{5.2.10}$$

5.3 MAXIMUM LIKELIHOOD ESTIMATION

The grouped data $\{x_{ij},\ j = 1,2,...,K(i)+1\}$ is multinomially distributed with parameters n_i and $\{P_{ij},\ j = 1,2,...,K(i)+1\}$ at the stress level s_i. The likelihood function is

$$L' = \prod_{i=1}^{\ell} L_i'$$

$$= \prod_{i=1}^{\ell} n_i! \left\{\prod_{j=1}^{K(i)+1} x_{ij}!\right\}^{-1} \left\{\prod_{j=1}^{K(i)+1} P_{ij}^{x_{ij}}\right\} \quad . \tag{5.3.1}$$

Logarithm of equation (5.3.1) gives

$$L = \ln L' = \sum_{i=1}^{\ell} \ln L_i'$$

$$= C + \sum_{i=1}^{\ell} \sum_{j=1}^{K(i)+1} x_{ij} \ln P_{ij} \ , \tag{5.3.2}$$

where C is constant and P_{ij} is defined as

$$\begin{aligned} P_{ij} &= F(t_{ij}) - F(t_{ij-1}) \\ &= \left(1 + (t_{ij-1}/\theta)^{\delta}\right)^{-m} - \left(1 + (t_{ij}/\theta)^{\delta}\right)^{-m} \ , \end{aligned} \tag{5.3.3}$$

for $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, K(i) + 1$.

Differentiating both sides of equation (5.3.2) with respect to β_0 and β_1 and equate it to zero. The ML estimates of β_0 and β_1 can be obtained by solving the following equations:

$$\begin{aligned} \frac{\partial L}{\partial \beta_0} &= 0 \\ &= \sum_{i=1}^{\ell} \sum_{j=1}^{K(i)+1} \frac{x_{ij}}{P_{ij}} \left[\frac{m\delta (t_{ij-1}/\theta)^{\delta}}{\left(1 + (t_{ij-1}/\theta)^{\delta}\right)^{(m+1)}} - \frac{m\delta (t_{ij}/\theta)^{\delta}}{\left(1 + (t_{ij}/\theta)^{\delta}\right)^{(m+1)}} \right] . \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \beta_1} &= 0 \\ &= \sum_{i=1}^{\ell} s_i \sum_{j=1}^{K(i)+1} \frac{x_{ij}}{P_{ij}} \left[\frac{m\delta (t_{ij-1}/\theta)^{\delta}}{\left(1 + (t_{ij-1}/\theta)^{\delta}\right)^{(m+1)}} - \frac{m\delta (t_{ij}/\theta)^{\delta}}{\left(1 + (t_{ij}/\theta)^{\delta}\right)^{(m+1)}} \right] \end{aligned}$$

The above equations can be rewritten as

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{\ell} \sum_{j=1}^{K(i)+1} \left(x_{ij} (A_{ij-1} - A_{ij}) / P_{ij}\right) = 0, \tag{5.3.4}$$

$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{\ell} s_i \sum_{j=1}^{K(i)+1} \left(x_{ij} (A_{ij-1} - A_{ij}) / P_{ij}\right) = 0, \tag{5.3.5}$$

where

$$A_{ij} = m\delta (t_{ij}/\theta)^{\delta} \left(1 + (t_{ij}/\theta)^{\delta}\right)^{-(m+1)}, \tag{5.3.6}$$

for $i = 1, 2, \ldots, \ell$ and $j = 0, 1, 2, \ldots, K(i) + 1$.

Fisher Information Matrix

The Fisher information matrix F, is obtained by taking negative s-expectations of second partial derivatives of L with respect to β_0 and β_1. That is,

$$F = \begin{bmatrix} \sum_{i=1}^{\ell} E\left\{-\frac{\partial^2 L}{\partial \beta_0^2}\right\} & \sum_{i=1}^{\ell} E\left\{-\frac{\partial^2 L}{\partial \beta_0 \partial \beta_1}\right\} \\ \sum_{i=1}^{\ell} E\left\{-\frac{\partial^2 L}{\partial \beta_1 \partial \beta_0}\right\} & \sum_{i=1}^{\ell} E\left\{-\frac{\partial^2 L}{\partial \beta_1^2}\right\} \end{bmatrix}$$

$$= N.\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix}, \tag{5.3.7}$$

where

$$f_{gh} = \sum_{i=1}^{\ell} \alpha_i \sum_{j=1}^{K(i)+1} \left(\frac{\partial P_{ij}}{\partial \beta_g}\right)\left(\frac{\partial P_{ij}}{\partial \beta_h}\right) / P_{ij}, \quad g, h = 0, 1. \tag{5.3.8}$$

After some algebraic manipulation, we obtain

$$f_{00} = \sum_{i=1}^{\ell} \alpha_i Q_i, \tag{5.3.9}$$

$$f_{01} = f_{10} = \sum_{i=1}^{\ell} \alpha_i s_i Q_i, \tag{5.3.10}$$

$$f_{11} = \sum_{i=1}^{\ell} \alpha_i s_i^2 Q_i \ , \tag{5.3.11}$$

where

$$Q_i = \sum_{j=1}^{K(i)+1} \left(A_{ij-1} - A_{ij}\right)^2 / P_{ij} \ . \tag{5.3.12}$$

Asymptotic Variance of ML estimates of the parameters

The asymptotic covariance matrix V of $\hat{\beta}_0$ and $\hat{\beta}_1$ is the inverse of the Fisher information matrix F.

$$V = F^{-1} = N^{-1} . \begin{bmatrix} \mathrm{var}(\hat{\beta}_0) & \mathrm{cov}(\hat{\beta}_0, \hat{\beta}_1) \\ \mathrm{cov}(\hat{\beta}_1, \hat{\beta}_0) & \mathrm{var}(\hat{\beta}_1) \end{bmatrix}$$

$$= N^{-1}\left(f_{00} f_{11} - f_1^2\right)^{-1} \begin{bmatrix} f_{11} & -f_{01} \\ -f_{10} & f_{00} \end{bmatrix} . \tag{5.3.13}$$

Then, the asymptotic variance of $\hat{y}_q$ is obtained as

$$\begin{aligned} \mathrm{AsVar}(\hat{y}_q) &= [1,\ s_0]\ F^{-1}\ [1,\ s_0]' \qquad (5.3.14) \\ &= N^{-1}(f_{00} f_{11} - f_{01}^2)^{-1}(f_{11} + s_0^2 f_{00} - 2 s_0 f_{01}) \ , \end{aligned}$$

which is also the asymptotic variance of $\hat{\mu}_0$.

Design Problem and Optimality Criterion

Optimal design problem of an ALT under periodic inspection can now be stated as: given N, s_0, s_1, δ, m $\{t_{ci},\ i = 1, 2, \ldots, \ell\}$ and $\{K(i),\ i = 1, 2, \ldots, \ell\}$, determine $\{\alpha_i,\ i = 1, 2, \ldots, \ell-1\}$, $\{s_i,\ i = 1, 2, \ldots, \ell-1\}$ and $\{t_{ij},\ i = 1, 2, \ldots, \ell;\ j = 1, 2, \ldots, K(i)-1\}$ such that the $\mathrm{AsVar}(\hat{y}_q)$ (or equivalently the $\mathrm{AsVar}(\hat{\mu}_0)$) is minimized.

We assume that the use stress level (s_0) is adjusted to be zero. Then, equation (5.3.14) becomes

$$\mathrm{AsVar}(\hat{y}_q) = N^{-1}(f_{00}f_{11} - f_{01}^2)^{-1}f_{11} \ . \qquad (5.3.15)$$

After some algebraic manipulation we obtain

$$\mathrm{AsVar}(\hat{y}_q) = N^{-1}(s_2^2Q_2 + (s_1^2Q_1 - s_2^2Q_2)\alpha_1)/ (Q_1Q_2(s_1 - s_2)^2(-\alpha_1^2 + \alpha_1)) \ . \qquad (5.3.16)$$

Minimizing the asymptotic variance of $\hat{y}_q$ of the lifetime distribution at design stress is used as the optimality criterion.

In section 5.4, we consider optimal ALT plans in which ℓ = 2, and determine s_1, $\alpha_1$and $\{t_{ij}$, i =1, 2; j = 1, 2, ... , K(i)-1$\}$ for various values of K(i).

5.4 OPTIMAL PLANS

Assumptions

1. Censoring times at s_1 and s_2 are the same.
   That is, $t_{C1} = t_{C2} = t_C$.
2. The number of inspections at each stress level is the same. That is, K(1) = K(2) = K. Further, we standardize the parameters such that $s_2 = 1$ as well as $t_C = 1$. In section 5.3, the use stress level was already adjusted to be 0. Such standardization does not alter the nature of our problem (see Appendix).

Optimization Method

The optimal plan is developed by determining optimal values of s_1 and α_1 for given N, K, δ and m such that $\mathrm{AsVar}(\hat{y}_q)$ is minimized. The two-step procedure is adopted for finding minimum value of $\mathrm{AsVar}(\hat{y}_q)$ with respect to s_1 and α_1 (see Appendix). The optimization procedure is initiated by first providing "guesstimates" of P_u and P_h which are used instead of β_0 and β_1.

P_u = Probability that test unit fails in $(0, t_c)$ at the use condition. (5.4.1)

P_h = Probability that a test unit fails in $(0, t_c)$ at the high stress. (5.4.2)

It is clear that P_u and P_h are more familiar and easier to estimate than β_0 and β_1. The corresponding β_0 and β_1 are obtained as follows.

$$\beta_0 = 1/\delta.\ln\left\{\frac{(1-P_u)^{1/m}}{1-(1-P_u)^{1/m}}\right\} . \tag{5.4.3}$$

$$\beta_1 = 1/\delta.\ln\left\{\frac{1-P_h}{1-P_u}\right\}^{1/m} + 1/\delta.\ln\left\{\frac{1-(1-P_u)^{1/m}}{1-(1-P_h)^{1/m}}\right\} . \tag{5.4.4}$$

The equally spaced inspection times are calculated at stress level s_i as

$$t_{ij} = j/K , \tag{5.4.5}$$

for $i = 1, 2$ and $j = 1, 2, \ldots, K$.

For given values of K, P_u, P_h, δ and m, optimal values of s_1 and α_1 are determined by two-step procedure such that $\mathrm{AsVar}(\hat{y}_q)$ is minimized. For $K = \infty$, the optimal values of s_1, α_1 and $\mathrm{AsVar}(\hat{y}_q)$ are determined using the method described by Nelson and Meeker (1978). It can also be determined by the two-step procedure. Further, ratio of $\mathrm{AsVar}(\hat{y}_q(K))$ to $\mathrm{AsVar}(\hat{y}_q(\infty))$ is obtained. The computer program on ALT, coded in FORTRAN is developed and run on VAX-11/780.

Sensitivity Analysis

For optimal ALT plans, we require the knowledge of P_u and P_h

(or equivalently β_0 and β_1). Chernoff (1953) termed this situation "locally optimal design" and suggested sensitivity analysis. The sensitivity analysis is conducted for some selected plausible values of P_u, P_h, δ and m, to see how AsVar($\hat{y}_q$) varies around these plausible values. Let $\tilde{P}_u$ and $\tilde{P}_h$ be the guessed values of P_u and P_h, respectively. s_1^* and α_1^* are determined using these guessed values with K = 2 as $\tilde{s}_1^*$ and $\tilde{\alpha}_1^*$, respectively. The sensitivity is defined as the ratio of AsVar($\hat{y}_q(\tilde{s}_1^*,\tilde{\alpha}_1^*)$) to AsVar($\hat{y}_q(s_1^*,\alpha_1^*)$) for various cases with K=2. Sensitivity analysis may be conducted for various values of K.

Sample size

For an optimal ALT plan, the desired precision of the MLE of the log mean life or q^{th} quantile with known shape parameters is determined by the sample size N^*. The required sample size (e.g., see Meeker (1986)) is approximately given by

$$N^* \cong \frac{\text{AsVar}(\hat{y}_q) W^2}{(\ln h)^2} \tag{5.4.6}$$

where W is the $(1 + \phi)/2$ quantile of the standard normal distribution.

5.5 COMPUTATIONAL RESULTS AND COMPARATIVE STUDY

The optimum plans are summarized in Tables 1-12, which present the optimum s_1^*, α_1^* and AsVar($\hat{y}_q$) for various combinations of P_u, P_h, K, δ and m. Tables 13-16 illustrate computational results of sensitivity analysis for various values of P_u, P_h and δ with m = 1 and K = 2. The Tables exhibit the following trends:

1. As P_u increases and/or P_h decreases, AsVar($\hat{y}_q$) is not sensitive to K for given values of shape parameters δ and m.

2. For selected values of P_u, P_h, δ and m, s_1^* and α_1^* are fairly stable over K.
3. For given δ and m, AsVar($\hat{y}_q$) decreases as P_u increases and it increases when P_h decreases.
4. For all the cases considered, the values of ratio indicate that the number of inspections (K) need not be too large, which is an encouraging result in terms of testing efforts.
5. For given δ and m, as P_u increases and/or P_h decreases the values of s_1^* and α_1^* tend to 0 (the use stress) and 1, respectively. This implies that almost no need for an ALT. Similar trends are observed when P_h values are small for $P_u < 0.1$.
6. For given values of m, as δ increases, asymptotic variance of $\hat{y}_q$ decreases.
7. For given δ, if the shape parameter m increases and P_u increases, then AsVar($\hat{y}_q$) increases except for $P_u = 0.1$.
8. For given δ and P_u, when shape parameter m increases, AsVar($\hat{y}_q$) decreases as P_h decreases.
9. For given values of δ and m, sensitivity values are fairly stable over K.
10. The sensitivity values are very close to 1, implying tha the AsVar($\hat{y}_q$) is robust against the true P_u and P_h form their guessed values, implying that the sample size may not be necessary to adjust.

5.6 PROCEDURES FOR PLANNING AN ALT

Based upon the above discussion, we suggest the following guidelines for planning an ALT.

1. Provide guesstimates of P_u and P_h and the ranges of their plausible values.
2. Determine the censoring time.

3. For various values of K, determine optimal plans with given δ and m.
4. For given δ and m, conduct a sensitivity analysis with respect to the plausible values of P_u and P_h, to see how $\mathrm{AsVar}(\hat{y}_q)$ varies around these plausible values.
5. Check the necessity of an ALT based upon s_1^* and α_1^*.
6. Determine the sample size.
7. Adjust N^* if necessary based upon the results of the sensitivity analysis.

5.7 CONCLUDING REMARKS

We have considered the problem of optimally designing ALT plans for the log-logistic and Burr type XII distributions under periodic inspection and Type I censoring.

We conclude that the log-logistic and Burr Type XII distributions are also used to yield an asymptotically optimal ALT plans. Since the trends obtained here are similar to the trends obtained by Yum and Choi (1989) and Seo and Yum (1991), we also conclude that the number of inspections (K) at stress level need not be too large and ALT plan is robust against the moderate departures of true P_u and P_h for their guessed values with known shape parameters δ and m. Equally spaced (ES) inspection times at each stress level are considered and optimal plans use only two stress levels in computational experiments. This scheme is administratively convenient and may be statistically optimal. Further work based on this method is needed to develop ALT plans for other models.

TABLE 1. OPTIMAL ALT PLANS WHEN δ = 0.5 AND m = 1.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	18.4201	-22.8146	1	0.596	0.647	775.4652	1.4161
				5	0.616	0.684	635.7888	1.1610
				10	0.622	0.697	596.9892	1.0902
				∞	0.632	0.715	547.6198	1
	.5	18.4201	-18.4201	1	0.664	0.787	1379.3949	1.0327
				5	0.668	0.793	1344.6050	1.0067
				10	0.668	0.795	1339.4281	1.0028
				∞	0.670	0.794	1335.6889	1
	.1	18.4201	-14.0257	1	0.624	0.845	5066.4662	1.0006
				5	0.624	0.845	5063.8171	1.0001
				10	0.624	0.845	5063.6003	1.0000
				∞	0.624	0.845	5063.4864	1
	.01	18.4201	-9.2299	1	0.444	0.889	21100.5332	1.0000
				5	0.444	0.889	21100.4593	1.0000
				10	0.444	0.889	21100.4539	1.0000
				∞	0.444	0.889	21100.4512	1
	.001	18.4201	-4.6066	1	0.002	0.999	40024.0281	1.0000
				5	0.002	0.999	40024.0276	1.0000
				10	0.002	0.999	40024.0276	1.0000
				∞	0.002	0.999	40024.0276	1
0.001	.9	13.8135	-18.2080	1	0.494	0.688	435.9018	1.3617
				5	0.520	0.718	365.5392	1.1419
				10	0.528	0.729	345.6581	1.0798
				∞	0.538	0.747	320.1218	1
	.5	13.8135	-13.8135	1	0.554	0.815	721.0074	1.0283
				5	0.558	0.821	705.2172	1.0058
				10	0.558	0.822	702.8597	1.0024
				∞	0.560	0.822	701.1697	1
	.1	13.8135	-9.4191	1	0.440	0.885	2080.3997	1.0004
				5	0.442	0.884	2079.5801	1.0000
				10	0.442	0.884	2079.5131	1.0000
				∞	0.442	0.884	2079.4779	1
	.01	13.8135	-4.6233	1	0.002	0.999	4006.7124	1.0000
				5	0.002	0.999	4006.7111	1.0000
				10	0.002	0.999	4006.7110	1.0000
				∞	0.002	0.999	4006.7109	1
0.01	.9	9.1902	-13.5847	1	0.322	0.772	192.7697	1.2612
				5	0.356	0.788	169.0355	1.1060
				10	0.366	0.796	162.0368	1.0602
				∞	0.380	0.807	152.8411	1
	.5	9.1902	-9.1902	1	0.328	0.822	273.0988	1.0184
				5	0.336	0.884	269.1716	1.0038
				10	0.336	0.885	268.5830	1.0016
				∞	0.338	0.884	268.1610	1
	.1	9.1902	-4.7958	1	0.002	0.999	404.2969	1.0000
				5	0.002	0.999	404.2837	1.0000
				10	0.002	0.999	404.2828	1.0000
				∞	0.002	0.999	404.2823	1
0.1	.9	4.3944	-8.7889	1	0.002	0.998	44.4885	1.0128
				5	0.006	0.995	44.2944	1.0084
				10	0.020	0.986	44.1978	1.0062
				∞	0.042	0.974	43.9242	1
	.7	4.3944	-6.0890	1	0.002	0.999	44.5227	1.0042
				5	0.002	0.999	44.3605	1.0006
				10	0.002	0.999	44.3453	1.0002
				∞	0.002	0.999	44.3360	1

TABLE 2. OPTIMAL ALT PLANS WHEN δ = 1. AND m = 1.

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	9.2101	-11.4073	1	0.596	0.647	193.8663	1.5039
				5	0.628	0.711	140.3335	1.0886
				10	0.634	0.724	132.8058	1.0302
				∞	0.636	0.732	128.9117	1
	.5	9.2101	-9.2101	1	0.664	0.787	344.8487	1.0334
				5	0.670	0.793	334.3100	1.0018
				10	0.670	0.794	333.8599	1.0004
				∞	0.670	0.794	333.7163	1
	.1	9.2101	-7.0128	1	0.624	0.845	1266.6165	1.0006
				5	0.624	0.845	1265.8973	1.0000
				10	0.624	0.845	1265.8747	1.0000
				∞	0.624	0.845	1265.8676	1
	.01	9.2101	-4.6150	1	0.444	0.889	5275.1333	1.0000
				5	0.444	0.889	5275.1135	1.0000
				10	0.444	0.889	5275.1129	1.0000
				∞	0.444	0.889	5275.1127	1
	.001	9.2101	-2.3033	1	0.002	0.999	10006.0070	1.0000
				5	0.002	0.999	10006.0068	1.0000
				10	0.002	0.999	10006.0068	1.0000
				∞	0.002	0.999	10006.0068	1
0.001	.9	6.9068	-9.1040	1	0.494	0.688	108.9754	1.4365
				5	0.534	0.743	81.8131	1.0785
				10	0.542	0.753	77.8971	1.0269
				∞	0.544	0.761	75.8599	1
	.5	6.9068	-6.9068	1	0.554	0.815	180.2519	1.0288
				5	0.558	0.823	175.4684	1.0015
				10	0.560	0.822	175.2642	1.0004
				∞	0.560	0.822	175.1988	1
	.1	6.9068	-4.7095	1	0.440	0.885	520.0999	1.0004
				5	0.442	0.884	519.8774	1.0000
				10	0.442	0.884	519.8704	1.0000
				∞	0.442	0.884	519.8682	1
	.01	6.9068	-2.3116	1	0.002	0.999	1001.6781	1.0000
				5	0.002	0.999	1001.6777	1.0000
				10	0.002	0.999	1001.6777	1.0000
				∞	0.002	0.999	1001.6777	1
0.01	.9	4.5951	-6.7923	1	0.322	0.772	48.1924	1.3140
				5	0.376	0.804	38.8593	1.0596
				10	0.386	0.811	37.4289	1.0206
				∞	0.390	0.816	36.6751	1
	.5	4.5951	-4.5951	1	0.328	0.882	68.2747	1.0188
				5	0.336	0.885	67.0847	1.0010
				10	0.338	0.884	67.0333	1.0002
				∞	0.338	0.885	67.0169	1
	.1	4.5951	-2.3979	1	0.002	0.999	101.0742	1.0000
				5	0.002	0.999	101.0707	1.0000
				10	0.002	0.999	101.0706	1.0000
				∞	0.002	0.999	101.0706	1
0.1	.9	2.1972	-4.3944	1	0.002	0.998	11.1221	1.0193
				5	0.036	0.977	11.0060	1.0086
				10	0.050	0.971	10.9485	1.0034
				∞	0.056	0.969	10.9116	1
	.7	2.1972	-3.0445	1	0.002	0.999	11.1307	1.0043
				5	0.002	0.999	11.0856	1.0002
				10	0.002	0.999	11.0840	1.0000
				∞	0.002	0.999	11.0835	1

TABLE 3. OPTIMAL ALT PLANS WHEN δ = 2. AND m = 1.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	4.6050	-5.7036	1	0.596	0.647	48.4666	1.5074
				5	0.636	0.727	32.6862	1.0166
				10	0.636	0.731	32.2960	1.0045
				∞	0.636	0.733	32.1529	1
	.5	4.6050	-4.6050	1	0.664	0.787	86.2122	1.0334
				5	0.670	0.794	83.5562	1.0015
				10	0.670	0.794	83.4594	1.0004
				∞	0.670	0.794	83.4287	1
	.1	4.6050	-3.5064	1	0.624	0.845	316.6541	1.0006
				5	0.624	0.845	316.4801	1.0000
				10	0.624	0.845	316.4701	1.0000
				∞	0.624	0.845	316.4670	1
	.01	4.6050	-2.3075	1	0.444	0.889	1318.7833	1.0000
				5	0.444	0.889	1318.7785	1.0000
				10	0.444	0.889	1318.7782	1.0000
				∞	0.444	0.889	1318.7782	1
	.001	4.6050	-1.1517	1	0.002	0.999	2501.5018	1.0000
				5	0.002	0.999	2501.5017	1.0000
				10	0.002	0.999	2501.5017	1.0000
				∞	0.002	0.999	2501.5017	1
0.001	.9	3.4534	-4.5520	1	0.494	0.688	27.2439	1.4395
				5	0.544	0.757	19.2058	1.0148
				10	0.544	0.761	19.0008	1.0040
				∞	0.546	0.760	18.9256	1
	.5	3.4534	-3.4534	1	0.554	0.815	45.0630	1.0288
				5	0.558	0.823	43.8577	1.0013
				10	0.560	0.822	43.8135	1.0003
				∞	0.560	0.822	43.7995	1
	.1	3.4534	-2.3548	1	0.440	0.885	130.0250	1.0004
				5	0.442	0.884	129.9712	1.0000
				10	0.442	0.884	129.9681	1.0000
				∞	0.442	0.884	129.9671	1
	.01	3.4534	-1.1558	1	0.002	0.999	250.4195	1.0000
				5	0.002	0.999	250.4194	1.0000
				10	0.002	0.999	250.4194	1.0000
				∞	0.002	0.999	250.4194	1
0.01	.9	2.2976	-3.3962	1	0.322	0.772	12.0481	1.3161
				5	0.388	0.814	9.2588	1.0114
				10	0.390	0.816	9.1824	1.0031
				∞	0.390	0.817	9.1541	1
	.5	2.2976	-2.2976	1	0.328	0.882	17.0687	1.0188
				5	0.338	0.884	16.7690	1.0009
				10	0.338	0.884	16.7577	1.0002
				∞	0.338	0.885	16.7542	1
	.1	2.2976	-1.1989	1	0.002	0.999	25.2686	1.0000
				5	0.002	0.999	25.2677	1.0000
				10	0.002	0.999	25.2677	1.0000
				∞	0.002	0.999	25.2676	1
0.1	.9	1.0986	-2.1972	1	0.002	0.998	2.7805	1.0196
				5	0.054	0.969	2.7333	1.0023
				10	0.056	0.969	2.7288	1.0006
				∞	0.058	0.968	2.7271	1
	.7	1.0986	-1.5223	1	0.002	0.999	2.7827	1.0043
				5	0.002	0.999	2.7717	1.0003
				10	0.002	0.999	2.7711	1.0001
				∞	0.002	0.999	2.7709	1

TABLE 4. OPTIMAL ALT PLANS WHEN δ = 3. AND m = 1.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	3.0700	-3.8024	1	0.596	0.647	21.5407	1.5074
				5	0.636	0.727	14.5227	1.0163
				10	0.636	0.731	14.3464	1.0040
				∞	0.636	0.733	14.2897	1
	.5	3.0700	-3.0700	1	0.664	0.787	38.3165	1.0333
				5	0.670	0.793	37.1708	1.0024
				10	0.670	0.794	37.1018	1.0006
				∞	0.670	0.794	37.0800	1
	.1	3.0700	-2.3376	1	0.624	0.845	140.7352	1.0006
				5	0.624	0.845	140.6629	1.0001
				10	0.624	0.845	140.6548	1.0000
				∞	0.624	0.845	140.6521	1
	.01	3.0700	-1.5383	1	0.444	0.889	586.1261	1.0000
				5	0.444	0.889	586.1242	1.0000
				10	0.444	0.889	586.1239	1.0000
				∞	0.444	0.889	586.1238	1
	.001	3.0700	-0.7678	1	0.002	0.999	1111.7791	1.0000
				5	0.002	0.999	1111.7790	1.0000
				10	0.002	0.999	1111.7790	1.0000
				∞	0.002	0.999	1111.7790	1
0.001	.9	2.3023	-3.0347	1	0.494	0.688	12.1084	1.4396
				5	0.542	0.759	8.5339	1.0146
				10	0.544	0.761	8.4411	1.0036
				∞	0.546	0.760	8.4111	1
	.5	2.3023	-2.3023	1	0.554	0.815	20.0280	1.0288
				5	0.558	0.823	19.5080	1.0021
				10	0.560	0.822	19.4767	1.0005
				∞	0.560	0.822	19.4667	1
	.1	2.3023	-1.5698	1	0.440	0.885	57.7889	1.0004
				5	0.442	0.884	57.7665	1.0001
				10	0.442	0.884	57.7640	1.0000
				∞	0.442	0.884	57.7632	1
	.01	2.3023	-0.7705	1	0.002	0.999	111.2976	1.0000
				5	0.002	0.999	111.2976	1.0000
				10	0.002	0.999	111.2976	1.0000
				∞	0.002	0.999	111.2976	1
0.01	.9	1.5317	-2.2641	1	0.322	0.772	5.3547	1.3162
				5	0.388	0.814	4.1146	1.0114
				10	0.390	0.816	4.0797	1.0028
				∞	0.390	0.817	4.0684	1
	.5	1.5317	-1.5317	1	0.328	0.882	7.5861	1.0188
				5	0.336	0.885	7.4569	1.0014
				10	0.338	0.884	7.4489	1.0003
				∞	0.338	0.885	7.4464	1
	.1	1.5317	-0.7993	1	0.002	0.999	11.2305	1.0000
				5	0.002	0.999	11.2301	1.0000
				10	0.002	0.999	11.2301	1.0000
				∞	0.002	0.999	11.2301	1
0.1	.9	0.7324	-1.4648	1	0.002	0.998	1.2358	1.0196
				5	0.054	0.969	1.2151	1.0025
				10	0.056	0.969	1.2128	1.0006
				∞	0.058	0.968	1.2121	1
	.7	0.7324	-1.0148	1	0.002	0.999	1.2367	1.0042
				5	0.002	0.999	1.2321	1.0005
				10	0.002	0.999	1.2317	1.0001
				∞	0.002	0.999	1.2315	1

TABLE 5. OPTIMAL ALT PLANS WHEN δ = 0.5 AND m = 2.

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	19.8065	-21.3489	1	0.634	0.678	831.6177	1.3642
				5	0.660	0.723	662.4729	1.0868
				10	0.664	0.734	634.3262	1.0406
				∞	0.668	0.744	609.5900	1
	.5	19.8065	-18.0438	1	0.686	0.814	1333.7367	1.0139
				5	0.688	0.817	1318.2556	1.0022
				10	0.688	0.818	1316.4945	1.0008
				∞	0.688	0.818	1315.4163	1
	.1	19.8065	-13.9724	1	0.630	0.850	5005.2878	1.0003
				5	0.630	0.850	5003.9674	1.0000
				10	0.630	0.850	5003.8650	1.0000
				∞	0.630	0.850	5003.8125	1
	.01	19.8065	-9.2250	1	0.446	0.889	21076.4578	1.0000
				5	0.446	0.889	21076.4165	1.0000
				10	0.446	0.889	21076.4135	1.0000
				∞	0.446	0.889	21076.4120	1
	.001	19.8065	-4.6063	1	0.002	0.999	40022.9892	1.0000
				5	0.002	0.999	40022.9889	1.0000
				10	0.002	0.999	40022.9889	1.0000
				∞	0.002	0.999	40022.9889	1
0.001	.9	15.2002	-16.7426	1	0.534	0.714	460.4681	1.3190
				5	0.566	0.753	376.0482	1.0772
				10	0.572	0.762	361.7372	1.0362
				∞	0.578	0.769	349.0932	1
	.5	15.2002	-13.4375	1	0.578	0.839	697.1685	1.0121
				5	0.580	0.842	690.0983	1.0019
				10	0.580	0.842	689.2978	1.0007
				∞	0.580	0.842	688.8079	1
	.1	15.2002	-9.3661	1	0.448	0.889	2058.7185	1.0002
				5	0.448	0.889	2058.3090	1.0000
				10	0.448	0.889	2058.2772	1.0000
				∞	0.448	0.889	2058.2610	1
	.01	15.2002	-4.6187	1	0.002	0.999	4004.4851	1.0000
				5	0.002	0.999	4004.4844	1.0000
				10	0.002	0.999	4004.4843	1.0000
				∞	0.002	0.999	4004.4843	1
0.01	.9	10.5816	-12.1239	1	0.356	0.789	197.6643	1.2329
				5	0.400	0.812	169.7398	1.0587
				10	0.408	0.818	164.7724	1.0277
				∞	0.416	0.823	160.3237	1
	.5	10.5816	-8.8188	1	0.358	0.893	264.4008	1.0082
				5	0.360	0.895	262.5929	1.0013
				10	0.362	0.895	262.3880	1.0005
				∞	0.362	0.895	262.2624	1
	.1	10.5816	-4.7474	1	0.002	0.999	402.2422	1.0000
				5	0.002	0.999	402.2349	1.0000
				10	0.002	0.999	402.2343	1.0000
				∞	0.002	0.999	402.2341	1
0.1	.9	5.8341	-7.3764	1	0.002	0.998	42.2306	1.0082
				5	0.014	0.992	42.0875	1.0048
				10	0.028	0.985	42.0029	1.0028
				∞	0.040	0.980	41.8863	1
	.7	5.8341	-5.4512	1	0.002	0.999	42.2468	1.0023
				5	0.002	0.999	42.1601	1.0003
				10	0.002	0.999	42.1529	1.0001
				∞	0.002	0.999	42.1489	1

TABLE 6. OPTIMAL ALT PLANS WHEN δ = 1. AND m = 2.

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	9.9033	-10.6744	1	0.634	0.678	207.9044	1.3823
				5	0.668	0.739	154.1652	1.0250
				10	0.670	0.744	151.3659	1.0064
				∞	0.670	0.746	150.4098	1
	.5	9.9033	-9.0219	1	0.686	0.814	333.4342	1.0141
				5	0.688	0.818	329.0084	1.0006
				10	0.688	0.818	328.8546	1.0001
				∞	0.688	0.818	328.8063	1
	.1	9.9033	-6.9862	1	0.630	0.850	1251.3219	1.0003
				5	0.630	0.850	1250.9659	1.0000
				10	0.630	0.850	1250.9548	1.0000
				∞	0.630	0.850	1250.9513	1
	.01	9.9033	-4.6125	1	0.446	0.889	5269.1145	1.0000
				5	0.446	0.889	5269.1034	1.0000
				10	0.446	0.889	5269.1030	1.0000
				∞	0.446	0.889	5269.1029	1
	.001	9.9033	-2.3032	1	0.002	0.999	10005.7473	1.0000
				5	0.002	0.999	10005.7471	1.0000
				10	0.002	0.999	10005.7471	1.0000
				∞	0.002	0.999	10005.7471	1
0.001	.9	7.6001	-8.3713	1	0.534	0.714	115.1170	1.3346
				5	0.576	0.767	88.1774	1.0223
				10	0.578	0.772	86.7423	1.0057
				∞	0.578	0.774	86.2537	1
	.5	7.6001	-6.7187	1	0.578	0.839	174.2921	1.0123
				5	0.580	0.842	172.2722	1.0005
				10	0.580	0.842	172.2023	1.0001
				∞	0.580	0.842	172.1803	1
	.1	7.6001	-4.6831	1	0.448	0.889	514.6796	1.0002
				5	0.448	0.889	514.5692	1.0000
				10	0.448	0.889	514.5658	1.0000
				∞	0.448	0.889	514.5647	1
	.01	7.6001	-2.3093	1	0.002	0.999	1001.1213	1.0000
				5	0.002	0.999	1001.1211	1.0000
				10	0.002	0.999	1001.1211	1.0000
				∞	0.002	0.999	1001.1211	1
0.01	.9	5.2908	-6.0619	1	0.356	0.789	49.4161	1.2441
				5	0.414	0.821	40.4013	1.0172
				10	0.418	0.824	39.8935	1.0044
				∞	0.418	0.825	39.7195	1
	.5	5.2908	-4.4094	1	0.358	0.893	66.1002	1.0082
				5	0.362	0.895	65.5838	1.0004
				10	0.362	0.895	65.5657	1.0001
				∞	0.362	0.895	65.5601	1
	.1	5.2908	-2.3737	1	0.002	0.999	100.5606	1.0000
				5	0.002	0.999	100.5586	1.0000
				10	0.002	0.999	100.5585	1.0000
				∞	0.002	0.999	100.5585	1
0.1	.9	2.9171	-3.6882	1	0.002	0.998	10.5577	1.0093
				5	0.036	0.981	10.4816	1.0021
				10	0.042	0.979	10.4659	1.0006
				∞	0.044	0.978	10.4601	1
	.7	2.9171	-2.7256	1	0.002	0.999	10.5617	1.0023
				5	0.002	0.999	10.5381	1.0001
				10	0.002	0.999	10.5373	1.0000
				∞	0.002	0.999	10.5371	1

TABLE 7. OPTIMAL ALT PLANS WHEN δ = 2. AND m = 2.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	4.9516	-5.3372	1	0.634	0.678	51.9761	1.3826
				5	0.668	0.745	37.9634	1.0099
				10	0.670	0.745	37.6825	1.0024
				∞	0.670	0.746	37.5922	1
	.5	4.9516	-4.5109	1	0.686	0.814	83.3585	1.0141
				5	0.688	0.818	82.2652	1.0008
				10	0.688	0.818	82.2170	1.0002
				∞	0.688	0.818	82.2018	1
	.1	4.9516	-3.4931	1	0.630	0.850	312.8305	1.0003
				5	0.630	0.850	312.7446	1.0000
				10	0.630	0.850	312.7395	1.0000
				∞	0.630	0.850	312.7379	1
	.01	4.9516	-2.3062	1	0.446	0.889	1317.2786	1.0000
				5	0.446	0.889	1317.2759	1.0000
				10	0.446	0.889	1317.2758	1.0000
				∞	0.446	0.889	1317.2757	1
	.001	4.9516	-1.1516	1	0.002	0.999	2501.4368	1.0000
				5	0.002	0.999	2501.4367	1.0000
				10	0.002	0.999	2501.4367	1.0000
				∞	0.002	0.999	2501.4367	1
0.001	.9	3.8001	-4.1856	1	0.534	0.714	28.7793	1.3350
				5	0.578	0.771	21.7487	1.0088
				10	0.578	0.773	21.6044	1.0021
				∞	0.578	0.774	21.5582	1
	.5	3.8001	-3.3594	1	0.578	0.839	43.5730	1.0123
				5	0.580	0.842	43.0741	1.0007
				10	0.580	0.842	43.0521	1.0002
				∞	0.580	0.842	43.0452	1
	.1	3.8001	-2.3415	1	0.448	0.889	128.6699	1.0002
				5	0.448	0.889	128.6433	1.0000
				10	0.448	0.889	128.6417	1.0000
				∞	0.448	0.889	128.6412	1
	.01	3.8001	-1.1547	1	0.002	0.999	250.2803	1.0000
				5	0.002	0.999	250.2803	1.0000
				10	0.002	0.999	250.2803	1.0000
				∞	0.002	0.999	250.2803	1
0.01	.9	2.6454	-3.0310	1	0.356	0.789	12.3540	1.2444
				5	0.416	0.824	9.9961	1.0069
				10	0.418	0.825	9.9446	1.0017
				∞	0.418	0.825	9.9280	1
	.5	2.6454	-2.2047	1	0.358	0.893	16.5250	1.0082
				5	0.362	0.895	16.3976	1.0005
				10	0.362	0.895	16.3918	1.0001
				∞	0.362	0.895	16.3900	1
	.1	2.6454	-1.1869	1	0.002	0.999	25.1401	1.0000
				5	0.002	0.999	25.1397	1.0000
				10	0.002	0.999	25.1396	1.0000
				∞	0.002	0.999	25.1396	1
0.1	.9	1.4585	-1.8441	1	0.002	0.998	2.6394	1.0094
				5	0.042	0.979	2.6176	1.0010
				10	0.044	0.978	2.6156	1.0003
				∞	0.044	0.978	2.6150	1
	.7	1.4585	-1.3628	1	0.002	0.999	2.6404	1.0023
				5	0.002	0.999	2.6347	1.0002
				10	0.002	0.999	2.6344	1.0000
				∞	0.002	0.999	2.6343	1

TABLE 8. OPTIMAL ALT PLANS WHEN δ = 3. AND m = 2.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	3.3011	-3.5581	1	0.634	0.678	23.1005	1.3826
				5	0.668	0.744	16.9200	1.0127
				10	0.670	0.745	16.7592	1.0030
				∞	0.670	0.746	16.7083	1
	.5	3.3011	-3.0073	1	0.686	0.814	37.0483	1.0141
				5	0.688	0.817	36.5835	1.0013
				10	0.688	0.818	36.5463	1.0003
				∞	0.688	0.818	36.5345	1
	.1	3.3011	-2.3287	1	0.630	0.850	139.0358	1.0003
				5	0.630	0.850	139.0003	1.0000
				10	0.630	0.850	138.9961	1.0000
				∞	0.630	0.850	138.9947	1
	.01	3.3011	-1.5375	1	0.446	0.889	585.4574	1.0000
				5	0.446	0.889	585.4563	1.0000
				10	0.446	0.889	585.4561	1.0000
				∞	0.446	0.889	585.4561	1
	.001	3.3011	-0.7677	1	0.002	0.999	1111.7502	1.0000
				5	0.002	0.999	1111.7502	1.0000
				10	0.002	0.999	1111.7502	1.0000
				∞	0.002	0.999	1111.7502	1
0.001	.9	2.5334	-2.7904	1	0.534	0.714	12.7908	1.3349
				5	0.578	0.770	9.6907	1.0114
				10	0.578	0.773	9.6078	1.0027
				∞	0.578	0.774	9.5818	1
	.5	2.5334	-2.2396	1	0.578	0.839	19.3658	1.0123
				5	0.580	0.842	19.1537	1.0012
				10	0.580	0.842	19.1367	1.0003
				∞	0.580	0.842	19.1313	1
	.1	2.5334	-1.5610	1	0.448	0.889	57.1866	1.0002
				5	0.448	0.889	57.1756	1.0000
				10	0.448	0.889	57.1743	1.0000
				∞	0.448	0.889	57.1739	1
	.01	2.5334	-0.7698	1	0.002	0.999	111.2357	1.0000
				5	0.002	0.999	111.2357	1.0000
				10	0.002	0.999	111.2357	1.0000
				∞	0.002	0.999	111.2357	1
0.01	.9	1.7636	-2.0206	1	0.356	0.789	5.4907	1.2443
				5	0.416	0.823	4.4516	1.0088
				10	0.418	0.825	4.4219	1.0021
				∞	0.418	0.825	4.4126	1
	.5	1.7636	-1.4698	1	0.358	0.893	7.3445	1.0082
				5	0.362	0.895	7.2903	1.0008
				10	0.362	0.895	7.2859	1.0002
				∞	0.362	0.895	7.2845	1
	.1	1.7636	-0.7912	1	0.002	0.999	11.1734	1.0000
				5	0.002	0.999	11.1732	1.0000
				10	0.002	0.999	11.1732	1.0000
				∞	0.002	0.999	11.1732	1
0.1	.9	0.9724	-1.2294	1	0.002	0.998	1.1731	1.0093
				5	0.040	0.980	1.1638	1.0014
				10	0.044	0.978	1.1626	1.0003
				∞	0.044	0.978	1.1622	1
	.7	0.9724	-0.9085	1	0.002	0.999	1.1735	1.0023
				5	0.002	0.999	1.1711	1.0003
				10	0.002	0.999	1.1709	1.0001
				∞	0.002	0.999	1.1708	1

TABLE 9. OPTIMAL ALT PLANS WHEN δ = 0.5 AND m = 3.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	20.6187	-20.9059	1	0.652	0.702	805.6543	1.2798
				5	0.674	0.743	664.6118	1.0557
				10	0.678	0.749	644.7894	1.0243
				∞	0.680	0.756	629.5225	1
	.5	20.6187	-17.9239	1	0.696	0.829	1295.6948	1.0086
				5	0.696	0.832	1286.1615	1.0012
				10	0.696	0.832	1285.1863	1.0004
				∞	0.696	0.832	1284.6193	1
	.1	20.6187	-13.9560	1	0.634	0.854	4950.0042	1.0002
				5	0.634	0.854	4949.0672	1.0000
				10	0.634	0.854	4948.9958	1.0000
				∞	0.634	0.854	4948.9595	1
	.01	20.6187	-9.2245	1	0.446	0.890	21059.6854	1.0000
				5	0.446	0.890	21059.6532	1.0000
				10	0.446	0.890	21059.6508	1.0000
				∞	0.446	0.890	21059.6497	1
	.001	20.6187	-4.6072	1	0.002	0.999	40046.4155	1.0000
				5	0.002	0.999	40046.4151	1.0000
				10	0.002	0.999	40046.4151	1.0000
				∞	0.002	0.999	40046.4151	1
0.001	.9	16.0115	-16.2987	1	0.554	0.734	446.5847	1.2467
				5	0.582	0.770	376.0565	1.0498
				10	0.588	0.775	365.9948	1.0217
				∞	0.590	0.781	358.2094	1
	.5	16.0115	-13.3167	1	0.590	0.852	678.7217	1.0076
				5	0.592	0.853	674.3368	1.0011
				10	0.592	0.853	673.8870	1.0004
				∞	0.592	0.853	673.6255	1
	.1	16.0115	-9.3488	1	0.452	0.892	2040.5696	1.0002
				5	0.452	0.892	2040.2785	1.0000
				10	0.452	0.892	2040.2563	1.0000
				∞	0.452	0.892	2040.2450	1
	.01	16.0115	-4.6173	1	0.002	0.999	4004.1219	1.0000
				5	0.002	0.999	4004.1213	1.0000
				10	0.002	0.999	4004.1212	1.0000
				∞	0.002	0.999	4004.1212	1
0.01	.9	11.3942	-11.6814	1	0.378	0.802	192.2435	1.1827
				5	0.418	0.823	168.7470	1.0382
				10	0.424	0.827	165.2603	1.0167
				∞	0.428	0.831	162.5408	1
	.5	11.3942	-8.6994	1	0.372	0.901	258.7940	1.0052
				5	0.376	0.901	257.6457	1.0007
				10	0.376	0.901	257.5280	1.0003
				∞	0.376	0.901	257.4598	1
	.1	11.3942	-4.7315	1	0.002	0.999	401.5521	1.0000
				5	0.002	0.999	401.5464	1.0000
				10	0.002	0.999	401.5460	1.0000
				∞	0.002	0.999	401.5457	1
0.1	.9	6.6627	-6.9500	1	0.002	0.999	41.4886	1.0069
				5	0.022	0.989	41.3469	1.0034
				10	0.032	0.984	41.2766	1.0017
				∞	0.040	0.981	41.2056	1
	.7	6.6627	-5.2515	1	0.002	0.999	41.5063	1.0018
				5	0.002	0.999	41.4401	1.0002
				10	0.002	0.999	41.4348	1.0001
				∞	0.002	0.999	41.4320	1

TABLE 10. OPTIMAL ALT PLANS WHEN δ = 1. AND m = 3.

P_u	P_h	β_0	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	10.3093	-10.4530	1	0.652	0.702	201.4136	1.2879
				5	0.680	0.753	158.6216	1.0142
				10	0.682	0.755	156.9386	1.0035
				∞	0.682	0.756	156.3944	1
	.5	10.3093	-8.9620	1	0.696	0.829	323.9237	1.0087
				5	0.696	0.832	321.2479	1.0004
				10	0.696	0.832	321.1592	1.0001
				∞	0.696	0.832	321.1314	1
	.1	10.3093	-6.9780	1	0.634	0.854	1237.5011	1.0002
				5	0.634	0.854	1237.2490	1.0000
				10	0.634	0.854	1237.2411	1.0000
				∞	0.634	0.854	1237.2386	1
	.01	10.3093	-4.6123	1	0.446	0.890	5264.9213	1.0000
				5	0.446	0.890	5264.9127	1.0000
				10	0.446	0.890	5264.9124	1.0000
				∞	0.446	0.890	5264.9123	1
	.001	10.3093	-2.3036	1	0.002	0.999	10011.6039	1.0000
				5	0.002	0.999	10011.6037	1.0000
				10	0.002	0.999	10011.6037	1.0000
				∞	0.002	0.999	10011.6037	1
0.001	.9	8.0057	-8.1494	1	0.554	0.734	111.6462	1.2538
				5	0.590	0.778	90.1868	1.0128
				10	0.592	0.780	89.3275	1.0031
				∞	0.592	0.781	89.0495	1
	.5	8.0057	-6.6584	1	0.590	0.852	169.6804	1.0076
				5	0.592	0.853	168.4494	1.0003
				10	0.592	0.853	168.4084	1.0001
				∞	0.592	0.853	168.3956	1
	.1	8.0057	-4.6744	1	0.452	0.892	510.1424	1.0002
				5	0.452	0.892	510.0641	1.0000
				10	0.452	0.892	510.0616	1.0000
				∞	0.452	0.892	510.0609	1
	.01	8.0057	-2.3087	1	0.002	0.999	1001.0305	1.0000
				5	0.002	0.999	1001.0303	1.0000
				10	0.002	0.999	1001.0303	1.0000
				∞	0.002	0.999	1001.0303	1
0.01	.9	5.6971	-5.8407	1	0.378	0.802	48.0609	1.1879
				5	0.428	0.829	40.8580	1.0099
				10	0.430	0.830	40.5562	1.0024
				∞	0.430	0.831	40.4586	1
	.5	5.6971	-4.3497	1	0.372	0.901	64.6985	1.0052
				5	0.376	0.901	64.3763	1.0002
				10	0.376	0.901	64.3655	1.0001
				∞	0.376	0.901	64.3621	1
	.1	5.6971	-2.3657	1	0.002	0.999	100.3880	1.0000
				5	0.002	0.999	100.3865	1.0000
				10	0.002	0.999	100.3864	1.0000
				∞	0.002	0.999	100.3864	1
0.1	.9	3.3314	-3.4750	1	0.002	0.999	10.3721	1.0074
				5	0.038	0.982	10.3081	1.0012
				10	0.040	0.981	10.2993	1.0003
				∞	0.042	0.981	10.2962	1
	.7	3.3314	-2.6257	1	0.002	0.999	10.3766	1.0018
				5	0.002	0.999	10.3587	1.0001
				10	0.002	0.999	10.3581	1.0000
				∞	0.002	0.999	10.3579	1

TABLE 11. OPTIMAL ALT PLANS WHEN δ = 2. AND m = 3.

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	5.1547	-5.2265	1	0.652	0.702	50.3534	1.2880
				5	0.680	0.756	39.4134	1.0082
				10	0.682	0.756	39.1714	1.0020
				∞	0.682	0.756	39.0946	1
	.5	5.1547	-4.4810	1	0.696	0.829	80.9809	1.0087
				5	0.696	0.832	80.3243	1.0005
				10	0.696	0.832	80.2930	1.0001
				∞	0.696	0.832	80.2831	1
	.1	5.1547	-3.4890	1	0.634	0.854	309.3753	1.0002
				5	0.634	0.854	309.3145	1.0000
				10	0.634	0.854	309.3109	1.0000
				∞	0.634	0.854	309.3097	1
	.01	5.1547	-2.3061	1	0.446	0.890	1316.2303	1.0000
				5	0.446	0.890	1316.2282	1.0000
				10	0.446	0.890	1316.2281	1.0000
				∞	0.446	0.890	1316.2281	1
	.001	5.1547	-1.1518	1	0.002	0.999	2502.9010	1.0000
				5	0.002	0.999	2502.9009	1.0000
				10	0.002	0.999	2502.9009	1.0000
				∞	0.002	0.999	2502.9009	1
0.001	.9	4.0029	-4.0747	1	0.554	0.734	27.9115	1.2539
				5	0.590	0.781	22.4233	1.0073
				10	0.592	0.781	22.2997	1.0018
				∞	0.592	0.782	22.2603	1
	.5	4.0029	-3.3292	1	0.590	0.852	42.4201	1.0076
				5	0.592	0.853	42.1181	1.0005
				10	0.592	0.853	42.1036	1.0001
				∞	0.592	0.853	42.0990	1
	.1	4.0029	-2.3372	1	0.452	0.892	127.5356	1.0002
				5	0.452	0.892	127.5167	1.0000
				10	0.452	0.892	127.5156	1.0000
				∞	0.452	0.892	127.5152	1
	.01	4.0029	-1.1543	1	0.002	0.999	250.2576	1.0000
				5	0.002	0.999	250.2576	1.0000
				10	0.002	0.999	250.2576	1.0000
				∞	0.002	0.999	250.2576	1
0.01	.9	2.8485	-2.9204	1	0.378	0.802	12.0152	1.1880
				5	0.428	0.831	10.1715	1.0057
				10	0.430	0.831	10.1278	1.0014
				∞	0.430	0.831	10.1139	1
	.5	2.8485	-2.1749	1	0.372	0.901	16.1746	1.0052
				5	0.376	0.901	16.0956	1.0003
				10	0.376	0.901	16.0918	1.0001
				∞	0.376	0.901	16.0906	1
	.1	2.8485	-1.1829	1	0.002	0.999	25.0970	1.0000
				5	0.002	0.999	25.0966	1.0000
				10	0.002	0.999	25.0966	1.0000
				∞	0.002	0.999	25.0966	1
0.1	.9	1.6657	-1.7375	1	0.002	0.999	2.5930	1.0074
				5	0.040	0.981	2.5760	1.0008
				10	0.042	0.981	2.5745	1.0002
				∞	0.042	0.981	2.5740	1
	.7	1.6657	-1.3129	1	0.002	0.999	2.5941	1.0018
				5	0.002	0.999	2.5898	1.0001
				10	0.002	0.999	2.5896	1.0000
				∞	0.002	0.999	2.5895	1

TABLE 12. OPTIMAL ALT PLANS WHEN δ = 3. AND m = 3.

P_u	P_h	β_o	β_1	K	S_1^*	α_1^*	N AsVar($\hat{y}_q$)	RATIO
0.0001	.9	3.4364	-3.4843	1	0.652	0.702	22.3793	1.2879
				5	0.680	0.754	17.5795	1.0117
				10	0.682	0.755	17.4250	1.0028
				∞	0.682	0.756	17.3763	1
	.5	3.4364	-2.9873	1	0.696	0.829	35.9915	1.0087
				5	0.696	0.832	35.7142	1.0009
				10	0.696	0.832	35.6895	1.0002
				∞	0.696	0.832	35.6816	1
	.1	3.4364	-2.3260	1	0.634	0.854	137.5001	1.0002
				5	0.634	0.854	137.4750	1.0000
				10	0.634	0.854	137.4720	1.0000
				∞	0.634	0.854	137.4710	1
	.01	3.4364	-1.5374	1	0.446	0.890	584.9914	1.0000
				5	0.446	0.890	584.9905	1.0000
				10	0.446	0.890	584.9904	1.0000
				∞	0.446	0.890	584.9903	1
	.001	3.4364	-0.7679	1	0.002	0.999	1112.4007	1.0000
				5	0.002	0.999	1112.4006	1.0000
				10	0.002	0.999	1112.4006	1.0000
				∞	0.002	0.999	1112.4006	1
0.001	.9	2.6686	-2.7165	1	0.554	0.734	12.4051	1.2538
				5	0.590	0.779	9.9979	1.0105
				10	0.592	0.781	9.9189	1.0025
				∞	0.592	0.782	9.8940	1
	.5	2.6686	-2.2195	1	0.590	0.852	18.8534	1.0076
				5	0.592	0.853	18.7259	1.0008
				10	0.592	0.853	18.7144	1.0002
				∞	0.592	0.853	18.7108	1
	.1	2.6686	-1.5581	1	0.452	0.892	56.6825	1.0002
				5	0.452	0.892	56.6747	1.0000
				10	0.452	0.892	56.6738	1.0000
				∞	0.452	0.892	56.6734	1
	.01	2.6686	-0.7696	1	0.002	0.999	111.2256	1.0000
				5	0.002	0.999	111.2256	1.0000
				10	0.002	0.999	111.2256	1.0000
				∞	0.002	0.999	111.2256	1
0.01	.9	1.8990	-1.9469	1	0.378	0.802	5.3401	1.1879
				5	0.428	0.830	4.5320	1.0082
				10	0.430	0.831	4.5041	1.0020
				∞	0.430	0.831	4.4953	1
	.5	1.8990	-1.4499	1	0.372	0.901	7.1887	1.0052
				5	0.376	0.901	7.1554	1.0006
				10	0.376	0.901	7.1524	1.0001
				∞	0.376	0.901	7.1514	1
	.1	1.8990	-0.7886	1	0.002	0.999	11.1542	1.0000
				5	0.002	0.999	11.1541	1.0000
				10	0.002	0.999	11.1541	1.0000
				∞	0.002	0.999	11.1540	1
0.1	.9	1.1105	-1.1583	1	0.002	0.999	1.1525	1.0074
				5	0.038	0.982	1.1454	1.0012
				10	0.042	0.980	1.1444	1.0003
				∞	0.042	0.981	1.1440	1
	.7	1.1105	-0.8752	1	0.002	0.999	1.1530	1.0018
				5	0.002	0.999	1.1512	1.0002
				10	0.002	0.999	1.1509	1.0001
				∞	0.002	0.999	1.1509	1

TABLE 13. Sensitivities of AsVar($\hat{y}_q$) When δ = 0.5, and m=1.

P_u	P_h	$\tilde{P}_u \setminus \tilde{P}_h$					
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.6610	1.4088	1.1244	1.0285	1.3936
		0.00003	1.4113	1.2217	1.0388	1.0196	1.5988
		0.00005	1.3032	1.1482	1.0128	1.0341	1.7314
		0.00010	1.1875	1.0748	1	1.0774	1.9797
		0.00020	1.0996	1.0294	1.0178	1.1606	2.3078
		0.00030	1.0661	1.0185	1.0463	1.2388	2.5703
		0.00050	1.0457	1.0274	1.1077	1.3650	2.9885
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0929	1.1140	1.1311	1.1432	1.1554
		0.00003	1.0171	1.0302	1.0412	1.0505	1.0573
		0.00005	1.0019	1.0083	1.0142	1.0216	1.0260
		0.00010	1.0077	1.0012	1	1.0010	1.0026
		0.00020	1.0622	1.0326	1.0186	1.0110	1.0064
		0.00030	1.1274	1.0798	1.0542	1.0361	1.0273
		0.00050	1.2654	1.1827	1.1337	1.1004	1.0787
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1703	1.2219	1.2625	1.2933	1.3255
		0.00003	1.0319	1.0587	1.0852	1.1069	1.1310
		0.00005	1.0025	1.0151	1.0301	1.0464	1.0638
		0.00010	1.0228	1.0041	1	1.0025	1.0078
		0.00020	1.1738	1.0887	1.0461	1.0229	1.0108
		0.00030	1.3936	1.2212	1.1383	1.0888	1.0576
		0.00050	1.8871	1.5964	1.3937	1.2744	1.2024
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5756	1.7435	1.8868	2.0167	2.1244
		0.00003	1.0936	1.1912	1.2785	1.3607	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.7569	1.4933	1.1955	1.0640	1.2459
		0.00030	1.4097	1.2383	1.0606	1.0170	1.3860
		0.00050	1.2768	1.1490	1.0222	1.0209	1.4902
		0.00100	1.1392	1.0573	1	1.0589	1.6901
		0.00200	1.0556	1.0174	1.0282	1.1570	1.9839
		0.00300	1.0427	1.0266	1.0798	1.2502	2.2196
		0.00500	1.0834	1.0921	1.1942	1.4232	2.6135

Table 13. Cont...

	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1944	1.1932	1.1712	1.1442	1.1126
		0.00030	1.0547	1.0596	1.0550	1.0457	1.0343
		0.00050	1.0184	1.0204	1.0193	1.0166	1.0138
		0.00100	1.0052	1.0010	1	1.0016	1.0098
		0.00200	1.0594	1.0345	1.0264	1.0280	1.0419
		0.00300	1.1445	1.0929	1.0729	1.0708	1.0851
		0.00500	1.3477	1.2370	1.1870	1.1669	1.1789

	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1825	1.2291	1.2622	1.2969	1.3194
		0.00030	1.0336	1.0607	1.0841	1.1078	1.1234
		0.00050	1.0033	1.0156	1.0306	1.0465	1.0608
		0.00100	1.0223	1.0037	1	1.0023	1.0066
		0.00200	1.1742	1.0877	1.0466	1.0245	1.0120
		0.00300	1.3958	1.2233	1.1388	1.0887	1.0593
		0.00500	1.9161	1.5985	1.3914	1.2748	1.2019

	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5773	1.7455	1.8888	2.0186	2.1260
		0.00030	1.0937	1.1914	1.2788	1.3611	1.4373
		0.00050	1.0000	1.0372	1.0936	1.1529	1.2077
		0.00100	1.0000	1.0000	1	1.0000	1.0119
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

0.01	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00100	1.9539	1.6934	1.3586	1.1774	1.1528
		0.00300	1.4129	1.2833	1.1123	1.0413	1.1869
		0.00500	1.2274	1.1433	1.0407	1.0169	1.2366
		0.01000	1.0661	1.0302	1	1.0342	1.3536
		0.02000	1.0582	1.0391	1.0582	1.1393	1.5478
		0.03000	1.1812	1.1420	1.1675	1.2684	1.7150
		0.05000	1.6290	1.4722	1.4498	1.5508	1.0076

	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00100	1.3331	1.3692	1.3733	1.3525	1.3185
		0.00300	1.0723	1.1045	1.1210	1.1238	1.1179
		0.00500	1.0121	1.0298	1.0437	1.0526	1.0536
		0.01000	1.0316	1.0048	1	1.0023	1.0080
		0.02000	1.2895	1.1325	1.0675	1.0392	1.0314
		0.03000	1.4859	1.3633	1.2064	1.1334	1.0976
		0.05000	1.4862	1.4861	1.4860	1.4027	1.2836

	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00100	1.5882	1.7579	1.9003	2.0165	2.1064
		0.00300	1.0961	1.1937	1.2825	1.3653	1.4357
		0.00500	1.0000	1.0393	1.0973	1.1557	1.2111
		0.01000	1.0000	1.0000	1	1.0000	1.0160

Table 13. Cont...

		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0007	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	2.6969	2.4824	1.0664	1.7538	1.3987
		0.03000	1.4232	1.4271	1.3375	1.2455	1.1353
		0.05000	1.0922	1.1313	1.1219	1.0928	1.0439
		0.10000	1.0002	1.0001	1	1.0002	1.0022
		0.15000	1.0001	1.0000	1.0001	1.0004	1.0031
		0.20000	1.0000	1.0000	1.0001	1.0006	1.0038
		0.30000	1.0000	1.0001	1.0003	1.0010	1.0047
	0.7	$\tilde{P}_u \setminus \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.0407	2.1288	2.1391	2.0588	1.8745
		0.03000	1.1663	1.2611	1.3320	1.3607	1.3417
		0.05000	1.0021	1.0290	1.0921	1.1383	1.1570
		0.10000	1.0000	1.0000	1	1.0000	1.0002
		0.15000	1.0000	1.0000	1.0000	1.0001	1.0004
		0.20000	1.0001	1.0001	1.0001	1.0002	1.0005
		0.30000	1.0002	1.0002	1.0002	1.0004	1.0008

TABLE 14. Sensitivities of AsVar($\hat{y}_q$) When δ = 1., and m=1.

P_u	P_h						
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.5829	1.3674	1.1246	1.0281	1.3626
		0.00003	1.3439	1.1930	1.0388	1.0174	1.5456
		0.00005	1.2546	1.1305	1.0128	1.0292	1.6664
		0.00010	1.1510	1.0625	1	1.0687	1.8841
		0.00020	1.0750	1.0219	1.0155	1.1470	2.1971
		0.00030	1.0484	1.0152	1.0425	1.2147	2.4405
		0.00050	1.0374	1.0285	1.1072	1.3332	2.8060
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0928	1.1139	1.1311	1.1432	1.1554
		0.00003	1.0170	1.0301	1.0411	1.0505	1.0573
		0.00005	1.0019	1.0083	1.0142	1.0216	1.0259
		0.00010	1.0077	1.0013	1	1.0009	1.0026
		0.00020	1.0622	1.0326	1.0186	1.0110	1.0064
		0.00030	1.1275	1.0798	1.0542	1.0361	1.0273
		0.00050	1.2655	1.1827	1.1337	1.1004	1.0787
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1703	1.2219	1.2625	1.2933	1.3255
		0.00003	1.0319	1.0587	1.0852	1.1069	1.1310
		0.00005	1.0025	1.0151	1.0301	1.0464	1.0638

Table 14. Cont...

		0.00010	1.0228	1.0041	1	1.0025	1.0078
		0.00020	1.1738	1.0887	1.0461	1.0229	1.0108
		0.00030	1.3936	1.2212	1.1383	1.0888	1.0576
		0.00050	1.8871	1.5964	1.3937	1.2744	1.2024
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5756	1.7435	1.8868	2.0167	2.1244
		0.00003	1.0936	1.1912	1.2785	1.3607	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.6611	1.4491	1.1882	1.0648	1.2309
		0.00030	1.3459	1.2100	1.0604	1.0168	1.3562
		0.00050	1.2268	1.1276	1.0201	1.0181	1.4497
		0.00100	1.1068	1.0452	1	1.0521	1.6294
		0.00200	1.0405	1.0140	1.0281	1.1409	1.9017
		0.00300	1.0380	1.0271	1.0754	1.2288	2.1220
		0.00500	1.0929	1.0981	1.1869	1.3938	2.4771
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1868	1.1869	1.1728	1.1472	1.1205
		0.00030	1.0510	1.0565	1.0558	1.0471	1.0371
		0.00050	1.0164	1.0202	1.0198	1.0172	1.0143
		0.00100	1.0054	1.0010	1	1.0013	1.0073
		0.00200	1.0618	1.0363	1.0258	1.0265	1.0362
		0.00300	1.1485	1.0959	1.0720	1.0660	1.0774
		0.00500	1.3467	1.2361	1.1855	1.1639	1.1683
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1824	1.2290	1.2621	1.2968	1.3194
		0.00030	1.0336	1.0606	1.0840	1.1078	1.1234
		0.00050	1.0033	1.0156	1.0306	1.0464	1.0608
		0.00100	1.0223	1.0038	1	1.0023	1.0066
		0.00200	1.1742	1.0877	1.0467	1.0245	1.0120
		0.00300	1.3958	1.2234	1.1388	1.0887	1.0593
		0.00500	1.9162	1.5986	1.3915	1.2749	1.2020
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5773	1.7455	1.8888	2.0186	2.1260
		0.00030	1.0937	1.1914	1.2788	1.3611	1.4373
		0.00050	1.0000	1.0372	1.0936	1.1529	1.2077
		0.00100	1.0000	1.0000	1	1.0000	1.0119
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

Table 14. Cont...

		$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01	0.9	0.00100	1.8226	1.6205	1.3420	1.1750	1.1527
		0.00300	1.3409	1.2499	1.1086	1.0417	1.1761
		0.00500	1.1791	1.1215	1.0387	1.0158	1.2212
		0.01000	1.0465	1.0225	1	1.0290	1.3289
		0.02000	1.0608	1.0421	1.0544	1.1268	1.5160
		0.03000	1.1954	1.1459	1.1607	1.2461	1.6759
		0.05000	1.6607	1.4799	1.4300	1.5114	1.9632
	0.5	$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00100	1.3221	1.3591	1.3724	1.3614	1.3285
		0.00300	1.0682	1.1002	1.1206	1.1281	1.1221
		0.00500	1.0112	1.0291	1.0435	1.0528	1.0556
		0.01000	1.0317	1.0049	1	1.0025	1.0073
		0.02000	1.2945	1.1329	1.0676	1.0373	1.0262
		0.03000	1.4924	1.3638	1.2066	1.1273	1.0890
		0.05000	1.4926	1.4926	1.4925	1.3926	1.2703
	0.1	$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00100	1.5879	1.7577	1.9001	2.0162	2.1062
		0.00300	1.0961	1.1937	1.2824	1.3652	1.4356
		0.00500	1.0000	1.0393	1.0973	1.1557	1.2110
		0.01000	1.0000	1.0000	1	1.0000	1.0160
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0007	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	2.4237	2.2996	1.9922	1.7280	1.4036
		0.03000	1.3529	1.3789	1.3180	1.2424	1.1467
		0.05000	1.0727	1.1141	1.1178	1.0959	1.0611
		0.10000	1.0001	1.0001	1	1.0008	1.0018
		0.15000	1.0000	1.0000	1.0000	1.0003	1.0025
		0.20000	1.0000	1.0000	1.0001	1.0005	1.0031
		0.30000	1.0000	1.0001	1.0003	1.0008	1.0039
	0.7	$\tilde{P}_u \backslash \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.9781	2.0819	2.1021	2.0576	1.9068
		0.03000	1.1562	1.2563	1.3330	1.3751	1.3650
		0.05000	1.0001	1.0325	1.0967	1.1494	1.1776
		0.10000	1.0000	1.0000	1	1.0000	1.0001
		0.15000	1.0000	1.0000	1.0000	1.0001	1.0003
		0.20000	1.0001	1.0001	1.0001	1.0002	1.0004
		0.30000	1.0002	1.0002	1.0002	1.0003	1.0006

TABLE 15. Sensitivities of AsVar($\hat{y}_q$) When δ = 2., and m=1.

P_u	P_h	$\tilde{P}_u \setminus \tilde{P}_h$					
0.0001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.4391	1.2942	1.1159	1.0310	1.2764
		0.00003	1.2397	1.1423	1.0374	1.0112	1.4214
		0.00005	1.1677	1.0899	1.0118	1.0179	1.5139
		0.00010	1.0874	1.0360	1	1.0486	1.6926
		0.00020	1.0353	1.0102	1.0145	1.1116	1.9409
		0.00030	1.0225	1.0110	1.0407	1.1705	2.1273
		0.00050	1.0305	1.0344	1.0987	1.2758	2.4277
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0929	1.1140	1.1311	1.1432	1.1554
		0.00003	1.0171	1.0302	1.0412	1.0505	1.0573
		0.00005	1.0019	1.0083	1.0142	1.0216	1.0260
		0.00010	1.0077	1.0012	1	1.0010	1.0026
		0.00020	1.0622	1.0326	1.0186	1.0110	1.0064
		0.00030	1.1274	1.0798	1.0542	1.0361	1.0273
		0.00050	1.2654	1.1827	1.1337	1.1004	1.0787
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00001	1.1703	1.2219	1.2625	1.2933	1.3255
		0.00003	1.0319	1.0587	1.0852	1.1069	1.1310
		0.00005	1.0025	1.0151	1.0301	1.0464	1.0638
		0.00010	1.0228	1.0041	1	1.0025	1.0078
		0.00020	1.1738	1.0887	1.0461	1.0229	1.0108
		0.00030	1.3936	1.2212	1.1383	1.0888	1.0576
		0.00050	1.8871	1.5964	1.3937	1.2744	1.2024
	0.001	$\tilde{P}_u \setminus \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5756	1.7435	1.8868	2.0167	2.1244
		0.00003	1.0936	1.1912	1.2785	1.3607	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.4976	1.3586	1.1832	1.0709	1.1785
		0.00030	1.2379	1.1584	1.0574	1.0156	1.2718
		0.00050	1.1435	1.0888	1.0203	1.0112	1.3463
		0.00100	1.0550	1.0255	1	1.0354	1.4953
		0.00200	1.0200	1.0105	1.0255	1.1081	1.7118

Table 15. Cont...

		0.00300	1.0353	1.0350	1.0710	1.1804	1.8886
		0.00500	1.1151	1.1118	1.1732	1.3273	2.1907
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1850	1.1854	1.1722	1.1537	1.1271
		0.00030	1.0501	1.0557	1.0555	1.0473	1.0392
		0.00050	1.0158	1.0197	1.0196	1.0172	1.0145
		0.00100	1.0054	1.0010	1	1.0010	1.0049
		0.00200	1.0623	1.0367	1.0261	1.0244	1.0297
		0.00300	1.1494	1.0966	1.0723	1.0653	1.0682
		0.00500	1.3482	1.2372	1.1861	1.1586	1.1552
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1825	1.2291	1.2622	1.2969	1.3195
		0.00030	1.0336	1.0607	1.0841	1.1078	1.1234
		0.00050	1.0033	1.0156	1.0306	1.0465	1.0608
		0.00100	1.0223	1.0037	1	1.0023	1.0066
		0.00200	1.1742	1.0877	1.0466	1.0245	1.0120
		0.00300	1.3958	1.2233	1.1388	1.0887	1.0593
		0.00500	1.9161	1.5985	1.3914	1.2748	1.2019
	0.01	$\tilde{P}_u \setminus \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5773	1.7455	1.8888	2.0186	2.1260
		0.00030	1.0937	1.1914	1.2788	1.3611	1.4373
		0.00050	1.0000	1.0372	1.0936	1.1529	1.2077
		0.00100	1.0000	1.0000	1	1.0000	1.0119
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003
0.01	0.9	$\tilde{P}_u \setminus \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00100	1.6184	1.5138	1.3236	1.1850	1.1332
		0.00300	1.2347	1.1859	1.1032	1.0480	1.1399
		0.00500	1.1085	1.0864	1.0380	1.0152	1.1719
		0.01000	1.0205	1.0103	1	1.0184	1.2603
		0.02000	1.0773	1.0496	1.0522	1.1007	1.4209
		0.03000	1.2404	1.1649	1.1522	1.2056	1.5636
		0.05000	1.7431	1.5077	1.4081	1.4468	1.8237
	0.5	$\tilde{P}_u \setminus \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00100	1.3196	1.3569	1.3714	1.3627	1.3393
		0.00300	1.0672	1.0992	1.1201	1.1284	1.1305
		0.00500	1.0108	1.0286	1.0432	1.0551	1.0601
		0.01000	1.0320	1.0050	1	1.0023	1.0081
		0.02000	1.2955	1.1335	1.0680	1.0356	1.0223
		0.03000	1.4936	1.3649	1.2073	1.1246	1.0829
		0.05000	1.4939	1.4938	1.4937	1.3882	1.2575
	0.1	$\tilde{P}_u \setminus \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00100	1.5882	1.7579	1.9004	2.0165	2.1064

Table 15. Cont...

		0.00300	1.0961	1.1937	1.2825	1.3653	1.4357
		0.00500	1.0000	1.0393	1.0974	1.1557	1.2111
		0.01000	1.0000	1.0000	1	1.0000	1.0160
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0007	1.0006	1.0004	1.0003	1.0003
0.1	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.01000	2.0827	2.0555	1.8952	1.7094	1.4145
		0.03000	1.2394	1.2962	1.2910	1.2438	1.1629
		0.05000	1.0391	1.0828	1.1081	1.0992	1.0814
		0.10000	1.0030	1.0030	1	1.0031	1.0096
		0.15000	1.0030	1.0030	1.0030	1.0032	1.0046
		0.20000	1.0030	1.0030	1.0031	1.0033	1.0050
		0.30000	1.0031	1.0031	1.0032	1.0035	1.0056
	0.7	$\tilde{P}_u \backslash \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
		0.01000	1.9218	2.0281	2.0975	2.0955	1.9915
		0.03000	1.1455	1.2462	1.3290	1.3987	1.4162
		0.05000	1.0001	1.0321	1.1000	1.1676	1.2141
		0.10000	1.0000	1.0000	1	1.0000	1.0199
		0.15000	1.0000	1.0000	1.0000	1.0000	1.0001
		0.20000	1.0001	1.0001	1.0001	1.0001	1.0002
		0.30000	1.0002	1.0002	1.0002	1.0002	1.0004

TABLE 16. Sensitivities of AsVar($\hat{y}_q$) When δ = 3., and m=1.

P_u	P_h						
0.0001	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00001	1.3493	1.2445	1.1143	1.0392	1.1746
		0.00003	1.1835	1.1083	1.0362	1.0069	1.2766
		0.00005	1.1158	1.0632	1.0130	1.0077	1.3521
		0.00010	1.0514	1.0214	1	1.0295	1.4867
		0.00020	1.0163	1.0043	1.0156	1.0825	1.6745
		0.00030	1.0128	1.0119	1.0426	1.1287	1.8235
		0.00050	1.0313	1.0443	1.0960	1.2216	2.0661
	0.1	$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00001	1.0930	1.1141	1.1312	1.1433	1.1555
		0.00003	1.0171	1.0302	1.0412	1.0505	1.0573
		0.00005	1.0019	1.0083	1.0143	1.0216	1.0260
		0.00010	1.0077	1.0012	1	1.0010	1.0026
		0.00020	1.0621	1.0326	1.0186	1.0110	1.0064
		0.00030	1.1273	1.0797	1.0541	1.0361	1.0273
		0.00050	1.2653	1.1826	1.1336	1.1003	1.0787
	0.01	$\tilde{P}_u \backslash \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140

Table 16. Cont...

		0.00001	1.1703	1.2219	1.2625	1.2933	1.3255
		0.00003	1.0319	1.0587	1.0852	1.1069	1.1310
		0.00005	1.0025	1.0151	1.0301	1.0464	1.0638
		0.00010	1.0228	1.0041	1	1.0025	1.0078
		0.00020	1.1738	1.0887	1.0461	1.0229	1.0108
		0.00030	1.3936	1.2212	1.1383	1.0888	1.0576
		0.00050	1.8871	1.5964	1.3937	1.2744	1.2024
	0.001	$\tilde{P}_u \backslash \tilde{P}_h$	0.0006	0.0008	0.0010	0.0012	0.0014
		0.00001	1.5756	1.7435	1.8868	2.0167	2.1244
		0.00003	1.0936	1.1912	1.2785	1.3607	1.4369
		0.00005	1.0000	1.0373	1.0936	1.1529	1.2076
		0.00010	1.0000	1.0000	1	1.0000	1.0119
		0.00020	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00030	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00050	1.0008	1.0006	1.0004	1.0003	1.0003
0.001	0.9	$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
		0.00010	1.4017	1.3112	1.1776	1.0822	1.1121
		0.00030	1.1758	1.1224	1.0539	1.0170	1.1729
		0.00050	1.1007	1.0624	1.0203	1.0056	1.2287
		0.00100	1.0297	1.0122	1	1.0200	1.3426
		0.00200	1.0137	1.0114	1.0258	1.0781	1.5112
		0.00300	1.0422	1.0417	1.0715	1.1414	1.6567
		0.00500	1.1412	1.1342	1.1741	1.2666	1.8969
	0.5	$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
		0.00010	1.1910	1.1904	1.1760	1.1509	1.1248
		0.00030	1.0529	1.0581	1.0544	1.0489	1.0381
		0.00050	1.0173	1.0195	1.0189	1.0165	1.0155
		0.00100	1.0052	1.0009	1	1.0011	1.0052
		0.00200	1.0603	1.0352	1.0268	1.0253	1.0306
		0.00300	1.1461	1.0942	1.0736	1.0669	1.0696
		0.00500	1.3502	1.2391	1.1832	1.1611	1.1574
	0.1	$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
		0.00010	1.1827	1.2292	1.2624	1.2970	1.3195
		0.00030	1.0337	1.0608	1.0841	1.1079	1.1234
		0.00050	1.0033	1.0157	1.0306	1.0465	1.0609
		0.00100	1.0222	1.0037	1	1.0023	1.0066
		0.00200	1.1741	1.0876	1.0466	1.0245	1.0120
		0.00300	1.3956	1.2232	1.1387	1.0886	1.0593
		0.00500	1.9159	1.5984	1.3913	1.2748	1.2019
	0.01	$\tilde{P}_u \backslash \tilde{P}_h$	0.0060	0.0080	0.0100	0.0120	0.0140
		0.00010	1.5773	1.7455	1.8888	2.0186	2.1260
		0.00030	1.0937	1.1914	1.2788	1.3611	1.4373
		0.00050	1.0000	1.0372	1.0936	1.1529	1.2077
		0.00100	1.0000	1.0000	1	1.0000	1.0119
		0.00200	1.0002	1.0001	1.0001	1.0000	1.0000
		0.00300	1.0004	1.0003	1.0002	1.0001	1.0001
		0.00500	1.0008	1.0006	1.0004	1.0003	1.0003

Table 16. Cont...

		$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.01	0.9	0.00100	1.5187	1.4417	1.3176	1.2048	1.1085
		0.00300	1.1745	1.1512	1.0993	1.0562	1.0900
		0.00500	1.0700	1.0608	1.0356	1.0173	1.1084
		0.01000	1.0094	1.0037	1	1.0086	1.1739
		0.02000	1.0951	1.0624	1.0500	1.0754	1.3019
		0.03000	1.2785	1.1909	1.1486	1.1706	1.4263
		0.05000	1.8304	1.5560	1.3943	1.3824	1.6547

		$\tilde{P}_u \backslash \tilde{P}_h$	0.3000	0.4000	0.5000	0.6000	0.7000
	0.5	0.00100	1.3286	1.3652	1.3712	1.3614	1.3456
		0.00300	1.0705	1.1027	1.1199	1.1279	1.1300
		0.00500	1.0113	1.0288	1.0450	1.0548	1.0599
		0.01000	1.0321	1.0050	1	1.0023	1.0081
		0.02000	1.2913	1.1337	1.0682	1.0374	1.0226
		0.03000	1.4882	1.3653	1.2077	1.1278	1.0833
		0.05000	1.4884	1.4883	1.4883	1.3939	1.2620

		$\tilde{P}_u \backslash \tilde{P}_h$	0.0600	0.0800	0.1000	0.1200	0.1400
	0.1	0.00100	1.5885	1.7584	1.9008	2.0169	2.1067
		0.00300	1.0962	1.1939	1.2826	1.3654	1.4359
		0.00500	1.0000	1.0393	1.0974	1.1558	1.2112
		0.01000	1.0000	1.0000	1	1.0000	1.0160
		0.02000	1.0002	1.0001	1.0001	1.0000	1.0000
		0.03000	1.0004	1.0003	1.0002	1.0001	1.0001
		0.05000	1.0007	1.0006	1.0004	1.0003	1.0003

		$\tilde{P}_u \backslash \tilde{P}_h$	0.7000	0.8000	0.9000	0.9500	0.9900
0.1	0.9	0.01000	1.9130	1.9232	1.8664	1.7455	1.4547
		0.03000	1.1841	1.2449	1.2826	1.2620	1.1824
		0.05000	1.0212	1.0609	1.1067	1.1144	1.0936
		0.10000	1.0067	1.0067	1	1.0050	1.0195
		0.15000	1.0067	1.0067	1.0067	1.0068	1.0076
		0.20000	1.0068	1.0068	1.0068	1.0069	1.0079
		0.30000	1.0069	1.0069	1.0069	1.0070	1.0083

		$\tilde{P}_u \backslash \tilde{P}_h$	0.5000	0.6000	0.7000	0.8000	0.9000
	0.7	0.01000	1.9448	2.0396	2.1111	2.1311	2.0882
		0.03000	1.1463	1.2434	1.3321	1.4117	1.4656
		0.05000	1.0001	1.0292	1.0973	1.1733	1.2474
		0.10000	1.0000	1.0000	1	1.0000	1.0311
		0.15000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.20000	1.0001	1.0001	1.0001	1.0001	1.0001
		0.30000	1.0002	1.0002	1.0002	1.0002	1.0002

APPENDIX

STANDARDIZATION OF PARAMETERS

Let us consider use stress $s_0 = 0$ and high stress $s_2 = 1$. Under this reparameterization, let, with and without the prime represent the original and standardized scale, respectively. It has the following transformation:

$$s = (s' - s'_0)/(s'_2 - s'_0) . \tag{5.8.1}$$

It can be written as

$$s' = s(s'_2 - s'_0) + s'_0 . \tag{5.8.2}$$

Next, consider the standardization of time. Let $t_{C1} = t_{C2} = t'_C$ in the original time scale. In the standardized scale, every parameter whose unit is time must be divided by t'_C. That is, $\theta = \theta'/t'_C$. Therefore, we have the following relationship:

$$\begin{aligned}\theta &= e^{(\beta'_0 + \beta'_1 s')} / t'_C \\ &= e^{(\beta'_0 + \beta'_1\{s(s'_2 - s'_0) + s'_0\})} e^{-\ln t'_C} \\ &= e^{\{(\beta'_0 + \beta'_1 s'_0 - \ln t'_C) + \beta'_1 s(s'_2 - s'_0)\}} .\end{aligned} \tag{5.8.3}$$

Since $\theta = e^{(\beta_0 + \beta_1 s)}$, we have

$$\beta_0 = \beta'_0 + \beta'_1 s'_0 - \ln t'_C , \tag{5.8.4}$$

$$\beta_1 = \beta'_1 (s'_2 - s'_0) . \tag{5.8.5}$$

or, equivalently,

$$\beta'_1 = \beta_1/(s'_2 - s'_0) , \tag{5.8.6}$$

and

$$\begin{aligned} \beta_0' &= \beta_0 + \ln t_c' - \beta_1' s_0' \\ &= \beta_0 + \ln t_c' - \beta_1 s_0'/(s_2' - s_0') . \end{aligned} \qquad (5.8.7)$$

From equations (5.2.9) and (5.8.4), one can see

$$\hat{y}_q = \hat{\beta}_0' + \hat{\beta}_1' s_0' + 1/\delta . \ln\left\{\frac{1-(1-q)^{1/m}}{(1-q)^{1/m}}\right\}$$

$$= \hat{\beta}_0 + \ln t_c' + 1/\delta . \ln\left\{\frac{1-(1-q)^{1/m}}{(1-q)^{1/m}}\right\} . \qquad (5.8.8)$$

Also

$$\hat{y}_q = \hat{\beta}_0 + 1/\delta . \ln\left\{\frac{1-(1-q)^{1/m}}{(1-q)^{1/m}}\right\} . \qquad (5.8.9)$$

It can be shown that

$$\mathrm{AsVar}(\hat{y}_q') = \mathrm{AsVar}(\hat{y}_q) .$$

Therefore, no generality is lost under the above standardization.

Two-step Procedure

The two-step procedure was adopted to optimize $\mathrm{AsVar}(\hat{y}_q)$ respect to s_1 and α_1. This technique is applied to our problem according to the following steps:

(1) We optimize α_1 (say α_1^*).

That is, from equation (5.3.16)

$$\frac{\partial(\mathrm{avar}(\hat{y}_q))}{\partial\alpha_1} = N^{-1}\{s_1^2 Q_1 - Q_2)\alpha_1^2 + 2Q_2\alpha_1 - Q_2\} / \{Q_1 Q_2 (s_1 - 1)^2 (-\alpha_1^2 + \alpha_1)^2\} = 0. \qquad (5.8.10)$$

The optimal value of α_1, for $0 < \alpha_1 < 1$, is given by

$$\alpha_1^* = \left(-Q_2 + \sqrt{s_1^2 Q_1 Q_2}\right) / \left(s_1^2 Q_1 - Q_2\right) . \qquad (5.8.11)$$

(2) We employ the grid search with respect to s_1 to optimize $\mathrm{AsVar}(\hat{y}_q)$. That is, α_1^* and the minimum of $\mathrm{AsVar}(\hat{y}_q)$ are determined on the grid s_1 = d, 2d, 3d, ... , where d is the grid size. Finally, minimum value of $\mathrm{AsVar}(\hat{y}_q)$ is determined by optimal s_1 (i.e., s_1^*) and the corresponding α_1^* among all the grid points considered.

In actual computational experiments, the above method was tried for a given set of P_u, P_h (and corresponding β_0, β_1), δ, m and K with 500 different grid points. Besides, the grid size for s_1 is set to 0.002. Finally, N is set to 1, the optimal solution consists of s_1^* (and the corresponding α_1^*) for which $\mathrm{AsVar}(\hat{y}_q)$ attains the minimum among all the grid points considered.

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